

# Buckling Load of Symmetric Planar Frames with Semi-rigid Joints Using Graph Theory

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*Abstract: The main aim of this paper is to extend the recently developed methods for calculating the buckling loads of planar symmetric frames to include the effect of semi-rigidity of the joints. This is achieved by decomposing a symmetric model into two submodels and then healing them in such a manner that the union of the eigenvalues of the healed submodels result in the eigenvalues of the entire model. Thus the critical load of the frame is obtained using the eigenvalues of its submodels.*

*Key words: symmetry, buckling load, semi-rigid frame, decomposition, healing, graph theory*

## 1. INTRODUCTION

Symmetry has been widely used in science and engineering, Hargittai [1], Gruber [2], Glockner [3], Zingoni [4], Zingoni et al. [5]. Many eigenvalue problems arise in many scientific and engineering problems, Livesley [6]), Jennings and McKeown [7], Bathe and Wilson [8]. While the basic mathematical ideas are independent of the size of matrices, the numerical determination of eigenvalues and eigenvectors becomes more complicated as the dimensions of matrices increase. Special methods are beneficial for efficient solution of such problems, especially when their corresponding matrices are highly sparse.

Methods are developed for decomposing and healing the graph models of structures, in order to calculate the eigenvalues of matrices and graph matrices with special patterns. The eigenvectors corresponding to such patterns for the symmetry of Form I, Form II and Form III are studied in references, Kaveh and Sayarinejad [9,10], and the applications to vibrating mass-spring systems and frame structures are developed in Kaveh and

Sayarinejad [11] and Kaveh and Salimbahrami [12], respectively. These forms are also applied to calculating the buckling load of symmetric mechanical systems [13,14].

The main aim of this paper is to extend the method developed in Ref. [14] for calculating the buckling loads of the frames with rigid joints to include the effect of semi-rigidity of the joints. This is achieved by decomposing a symmetric model into two submodels and then performing healing in such a manner that the union of the eigenvalues of the healed submodels result in the eigenvalues of the entire model. Thus the critical load of the frame is obtained using the eigenvalues of its submodels.

## 2. TRANSFORMATION OF MATRICES TO CANONICAL FORMS

In this section, an  $N \times N$  symmetric matrix  $[M]$  is considered with all entries being real. For three special canonical forms, the eigenvalues of  $[M]$  are obtained using the

properties of its submatrices.

**Canonical Form I:** In this case  $[M]$  has the following pattern,

$$[M] = \left[ \begin{array}{c|c} [A]_{n \times n} & [0]_{n \times n} \\ \hline [0]_{n \times n} & [A]_{n \times n} \end{array} \right]_{N \times N} \quad (1)$$

with  $N = 2n$ .

Considering the set of eigenvalues of the submatrix  $[A]$  as  $\{l(A)\}$ , the set of eigenvalues of  $[M]$  can be obtained as:

$$\{\lambda(M)\} = \{\lambda(A)\} \cup \{\lambda(A)\}. \quad (2)$$

Since  $\det(M) = \det(A) \times \det(A)$ , the above relation becomes obvious. The sign  $\cup$  simply indicates the collection of the eigenvalues of the submatrices.

**Canonical Form II:** For this case, matrix  $[M]$  can be decomposed into the following form:

$$[M] = \left[ \begin{array}{c|c} [A]_{n \times n} & [B]_{n \times n} \\ \hline [B]_{n \times n} & [A]_{n \times n} \end{array} \right]_{N \times N} \quad (3)$$

The eigenvalues of  $[M]$  can be calculated as,

$$\{\lambda(M)\} = \{\lambda(C)\} \cup \{\lambda(D)\}, \quad (4)$$

where:

$$[C] = [A] + [B] \text{ and } [D] = [A] - [B]. \quad (5)$$

$[C]$  and  $[D]$  are called condensed submatrices of  $[M]$ . The proof of this Form can be considered as the special case of the proof for Form III, and it is not repeated for brevity.

**Canonical Form III:** This form has a Form II submatrix augmented by some rows and columns as shown in the following:

$$[M] = \left[ \begin{array}{cccc} [A] & [B] & L_{11} & \dots & L_{1k} \\ & & L_{21} & \dots & L_{2k} \\ & & L_{n1} & \dots & L_{nk} \\ [B] & [A] & L_{11} & \dots & L_{1k} \\ & & L_{21} & \dots & L_{2k} \\ & & L_{n1} & \dots & L_{nk} \\ C(2n+1,1) & C(2n+1,2n) & C(2n+1,2n+1) & \dots & C(2n+1,2n+k) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z(2n+k,1) & Z(2n+k,2n) & Z(2n+k,2n+1) & \dots & Z(2n+k,2n+k) \end{array} \right] \quad (6)$$

Where  $[M]$  is a  $(2n+k) \times (2n+k)$  matrix, with a  $2n \times 2n$  submatrix with the pattern of Form II, and  $k$  augmented columns and rows. The entries of the augmented columns are repeated in the first and second block for each column, and all the entries of  $[M]$  are real numbers.

Now  $[D]$  is obtained as  $[D] = [A] - [B]$ , and  $[E]$  is constructed as the following:

$$[E] = \left[ \begin{array}{cccc} & & L_{11} & \dots & L_{1k} \\ & & L_{21} & \dots & L_{2k} \\ & & \vdots & \ddots & \vdots \\ [A+B] & & L_{n1} & \dots & L_{nk} \\ & & \vdots & \ddots & \vdots \\ C(2n+1,1)+C(2n+1,n+1) & & C(2n+1,2n+1) & \dots & C(2n+1,2n+k) \\ \vdots & & \vdots & \ddots & \vdots \\ Z(2n+k,1)+Z(2n+k,n+1) & & Z(2n+k,2n+1) & \dots & Z(2n+k,2n+k) \end{array} \right] \quad (7)$$

The set of eigenvalues for  $[M]$  is obtained as:

$$\{\lambda(M)\} = \{\lambda(D)\} \cup \{\lambda(E)\}. \quad (8)$$

More refined description of the above forms and the proof for their validity can be found in Refs. [9,10]. An excellent description of other canonical forms can be found in Ref. [15].

In order to benefit the symmetry of the structures, two different methods can be employed. In the first method, the stiffness matrix of the structure with an appropriate numbering of the DOFs is constructed and according to the rule of Section 2, the

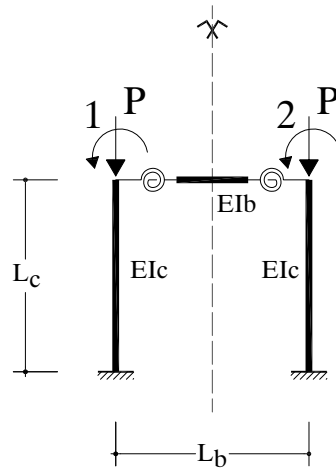


Fig. 1 A symmetric frame with two rotational degrees of freedom

submatrices C and D or D and E can be formed for Form II and Form III canonical forms, respectively. In the second method, the formation of the entire stiffness matrix can be avoided by decomposing and healing of the structural model [15]. Here, both methods are employed for studying frames with semi-rigid joints having odd and even numbers of spans for both non-sway and sway cases.

### 3. SYMMETRIC FRAMES WITH AN ODD NUMBER OF SPANS

Methods for including the effect of semi-rigidity of joints in frame structures are well-developed, Chen [16]. In Appendix a brief introduction is included and used for the formation of the stiffness matrices in subsequent sections.

#### 3.1 Frames without sway

In this section frames with an odd number of spans are studied. The symmetry axis of these frames passes through the beam elements and with an appropriate numbering of the DOFs, the stiffness matrix of this structure gain the Form II canonical form. As an example, the frame shown in Fig. 1 has

two rotational DOFs, and the corresponding stiffness matrix has the following form:

$$\mathbf{K} = \begin{bmatrix} \frac{4EI_c}{L_c} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} \\ \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{4EI_c}{L_c} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} \end{bmatrix} - \frac{P}{L_c} \begin{bmatrix} \frac{2L_c^2}{15} & 0 \\ 0 & \frac{2L_c^2}{15} \end{bmatrix} \quad (9)$$

In all the examples, b indicates that the element is a beam and c shows that the element is a column. Considering  $L_c = 200\text{cm}$ ,  $L_b = 100\text{cm}$ ,  $I = 100\text{cm}^4$ ,  $E = 2.01 \times 10^6 \text{Kg/cm}^2$  and  $k = 5 \times 10^6$  for the spring, the buckling load is calculated as

$$P_{cr} = 150 \text{ ton}$$

The stiffness matrix has the Form II symmetry and therefore using the results of Section 2 can be decomposed in order to find the stiffness matrices of its factors as:

$$\mathbf{K}_C = \left[ \frac{4EI_c}{L_c} + \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} \right] - \frac{2P}{15L_c} \quad (10a)$$

$$\mathbf{K}_D = \left[ \frac{4EI_c}{L_c} \right] - \frac{2P}{15L_c} \quad (10b)$$

In the following the factors C and D are obtained without the formation of the entire

stiffness matrix of the structure.

**Healing Process:**

For the non-sway frame with odd number of spans per storey and semi-rigid joints, healing consists of the following steps:

1. Delete the beams crossed by the axis of symmetry. These are link beams, identified by  $L_b$ . Now the structure is decomposed into two substructures  $S_1$  and  $S_2$  in the left and right hand sides, respectively.
2. For  $S_2$  add a hinge support to prevent the lateral displacement at the end of the column where the beam is cut.
3. For  $S_1$ , add a hinge support to prevent the lateral displacement at the end of the column where the beam is cut. Then add one rotational spring, with a stiffness equal to

$$\frac{12\beta EI_b + 8\gamma EI_b}{L_b^2}, \text{ to the joint at the } i^{\text{th}} \text{ storey.}$$

This provides the necessary stiffness requirement for obtaining the factor C.

$S_1$  and  $S_2$  are now healed and the factors C and D are obtained.

**Example 1:** Consider the frame shown in Fig. 1. Implementing the above process, the factors C and D are obtained as illustrated in

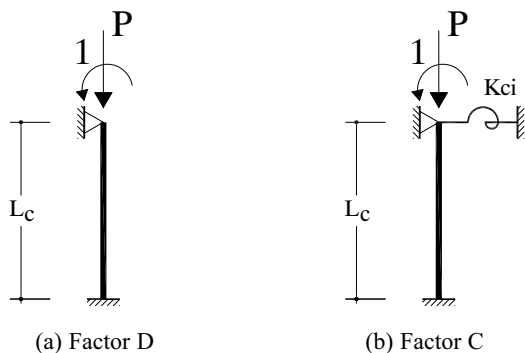


Fig. 2 Stiffness for the factors C and D

Fig. 2.

Now the stiffness matrices are formed for the factors as:

$$\mathbf{K}_C = \left[ \frac{4EI_C}{L_C} + \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} \right] - \frac{2P}{15L_C} \quad (11a)$$

$$\mathbf{K}_D = \left[ \frac{4EI_C}{L_C} \right] - \frac{2P}{15L_C} \quad (11b)$$

Using the same assumption as for the structure, the buckling loads of the substructures are calculated as:

$$(P_{c\gamma})_C = 370 \text{ ton}$$

$$(P_{c\gamma})_D = 150 \text{ ton}$$

and the smallest of this is 150 ton which is the buckling load of the structure.

**Example 2:** Now consider a two-story frame with four rotational DOFs as shown in Fig. 3.

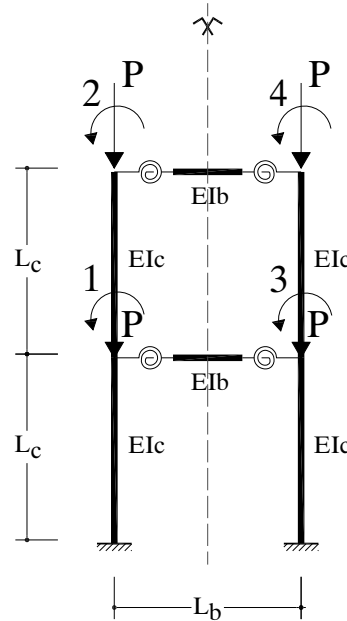


Fig. 3 A two-story frame with four rotational DOFs

The stiffness matrix is formed as:

$$\mathbf{K} = \begin{bmatrix} \frac{8EI_C + 6\beta EI_b + 4\gamma EI_b}{L_C} + \frac{2EI_C}{L_b} & \frac{2EI_C}{L_C} & \frac{6\beta EI_b + 4\gamma EI_b}{L_b} & 0 \\ \frac{2EI_C}{L_C} & \frac{4EI_C + 6\beta EI_b + 4\gamma EI_b}{L_C + L_b} & 0 & \frac{6\beta EI_b + 4\gamma EI_b}{L_b} \\ \frac{6\beta EI_b + 4\gamma EI_b}{L_b} & 0 & \frac{8EI_C + 6\beta EI_b + 4\gamma EI_b}{L_C} + \frac{2EI_C}{L_b} & \frac{2EI_C}{L_C} \\ 0 & \frac{6\beta EI_b + 4\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} & \frac{4EI_C + 6\beta EI_b + 4\gamma EI_b}{L_C + L_b} \end{bmatrix}$$

$$\frac{P}{L_C} \begin{bmatrix} \frac{4L_C^2}{15} - \frac{L_C^2}{30} & 0 & 0 \\ \frac{L_C^2}{30} & \frac{2L_C^2}{15} & 0 \\ 0 & 0 & \frac{4L_C^2}{15} - \frac{L_C^2}{30} \\ 0 & 0 & \frac{L_C^2}{30} & \frac{2L_C^2}{15} \end{bmatrix} \quad (12)$$

Considering  $L_C = 200\text{cm}$ ,  $L_b = 100\text{cm}$ ,  $I = 100\text{cm}^4$ ,  $E = 2.01 \times 10^6 \text{Kg/cm}^2$  and  $k=5 \times 10^6$  for the spring, the buckling load is calculated as

$$P_{cr} = 150 \text{ ton}$$

After healing, the factors C and D are obtained as shown in Fig. 4.

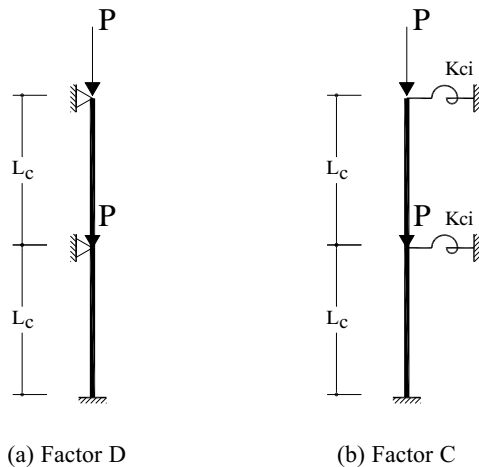


Fig. 4 Stiffness for the factors D and C

For the factors shown in Fig. 4, the stiffness and mass matrices are constructed in the follows:

$$\mathbf{K}_C = \begin{bmatrix} \frac{8EI_C + 12\beta EI_b + 8\gamma EI_b}{L_C} + \frac{2EI_C}{L_b} & \frac{2EI_C}{L_C} \\ \frac{2EI_C}{L_C} & \frac{4EI_C + 12\beta EI_b + 8\gamma EI_b}{L_C + L_b} \end{bmatrix} - \frac{P}{L_C} \begin{bmatrix} \frac{4L_C^2}{15} & -\frac{L_C^2}{30} \\ -\frac{L_C^2}{30} & \frac{2L_C^2}{15} \end{bmatrix} \quad (13a)$$

$$\mathbf{K}_D = \begin{bmatrix} \frac{8EI_C}{L_C} & \frac{2EI_C}{L_C} \\ \frac{2EI_C}{L_C} & \frac{4EI_C}{L_C} \end{bmatrix} - \frac{P}{L_C} \begin{bmatrix} \frac{4L_C^2}{15} & -\frac{L_C^2}{30} \\ -\frac{L_C^2}{30} & \frac{2L_C^2}{15} \end{bmatrix} \quad (13b)$$

Using the information from Fig. 3 and before decomposition, for the factors the buckling loads are calculated as:

$$\begin{aligned} (P_{cr})_C &= 150 \text{ ton} \\ (P_{cr})_D &= 37.48 \text{ ton} \end{aligned}$$

The smallest of these two, namely 37.48 ton, is the buckling load of the structure.

### 3.2 Frames with Sway

In this section the behaviour of the sway frames with semi-rigid joints is studied. For simplicity the axial deformation of the beams are neglected, and for each story level one translation DOF is considered. In order to have Form III symmetry, first the rotational DOFs are numbered suitable for Form II symmetry followed by story level translation DOFs. For the formation of the factors of the frame, a column is defined similar to that of Ref. [13].

The stiffness matrix of the column is as follows:

$$\mathbf{K} = \frac{2EI_C}{L_C^3} \begin{bmatrix} 6 & -6 & 3 & 3 \\ -6 & 6 & -3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \frac{P}{L_C} \begin{bmatrix} \frac{6}{5} & -\frac{6}{5} & \frac{L_C}{10} & \frac{L_C}{10} \\ -\frac{6}{5} & \frac{6}{5} & -\frac{L_C}{10} & -\frac{L_C}{10} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

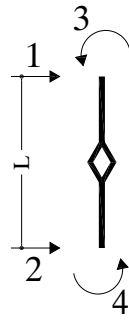


Fig. 5 A new column

**Healing Process:**

For symmetric frames with odd number of spans per storey and semi-rigid joints, healing consists of the following steps to obtain the factors D and E:

1. Delete all the beams which are crossed by the axis of symmetry. Now the structure is decomposed into two substructures  $S_1$  and  $S_2$  in the left and right hand sides, respectively.

2. For  $S_2$  the DOF of the beam are removed add a rotation spring with stiffness equal to

$$\frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b}$$

joint in each floor.

3. For  $S_1$ , a hinge support is added to prevent the lateral displacement at the end of the column where the beam is cut. This provides the necessary stiffness requirement for obtaining the factor D.

4. The effect of lateral displacement on E is provided by the addition of the new column, as shown in Fig. 5.

$S_1$  and  $S_2$  are now healed and the factors D and E are obtained.

**Example 3:** Consider a frame with two rotational DOFs and one translational DOF, as illustrated in Fig. 6. Nodal ordering is similar to the previously described process.

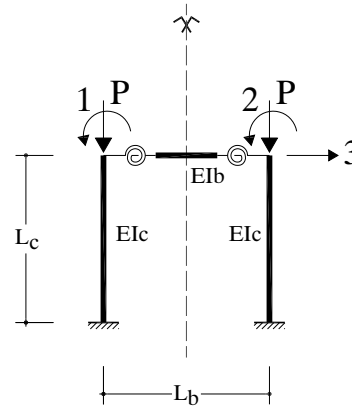


Fig. 6 A symmetric planar frame with sway

The stiffness matrix is constructed in the following:

$$K = \begin{bmatrix} \frac{4EI_c}{L_c} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & -\frac{6\beta EI_c}{L_c^2} \\ \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{4EI_c}{L_c} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & -\frac{6\beta EI_c}{L_c^2} \\ -\frac{6\beta EI_c}{L_c^2} & -\frac{6\beta EI_c}{L_c^2} & \frac{24\beta EI_c}{L_c^3} \end{bmatrix} - \frac{P}{L_c} \begin{bmatrix} \frac{2L_c^2}{15} & 0 & -\frac{L_c}{10} \\ 0 & \frac{2L_c^2}{15} & -\frac{L_c}{10} \\ -\frac{L_c}{10} & -\frac{L_c}{10} & \frac{12}{5} \end{bmatrix} \quad (15)$$

Considering  $L_c = 200\text{cm}$ ,  $L_b = 100\text{cm}$ ,  $I = 100\text{cm}^4$ ,  $E=2.01 \times 10^6 \text{Kg/cm}^2$  and  $k=5 \times 10^6$  for the spring, the buckling load is calculated as:

$$P_{cr} = 36.9 \text{ ton}$$

The above stiffness matrix has Form III symmetry. Therefore using the healing required for the Form III symmetry, the factors D and E are obtained, Fig. 7.

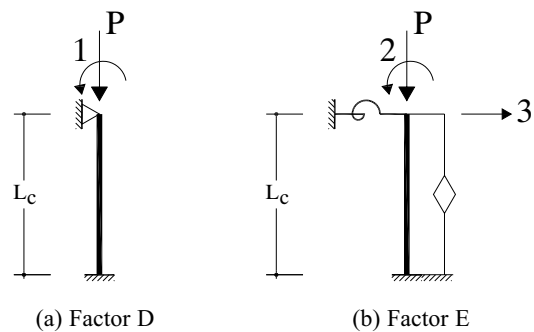


Fig. 7 The factors D and E of Fig. 6

The stiffness matrices of D and E are obtained as follows:

$$\mathbf{K}_D = \left[ \frac{4EI_C}{L_C} \right] - \frac{P}{L_C} \left[ \frac{2L_C^2}{15} \right] \quad (16a)$$

$$\mathbf{K}_E = \begin{bmatrix} \frac{4EI_C}{L_C} + \frac{12\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & -\frac{6\beta EI_C}{L_C^2} \\ -\frac{12\beta EI_C}{L_C^2} & \frac{24EI_C}{L_C^3} \end{bmatrix} - \frac{P}{L_C} \begin{bmatrix} \frac{2L_C^2}{15} & -\frac{L_C}{10} \\ -\frac{2L_C}{10} & \frac{12}{15} \end{bmatrix} \quad (16b)$$

Using the same data as before the decomposition, the buckling loads are calculated for the substructures as:

$$(P_{c\gamma})_D = 150 \text{ ton}$$

$$(P_{c\gamma})_E = 36.9 \text{ ton}$$

and the smallest of this is 36.9 ton which is the buckling load of the structure.

**Example 4:** Consider the two-story sway frame with semi-rigid joints, as shown in Fig. 8.

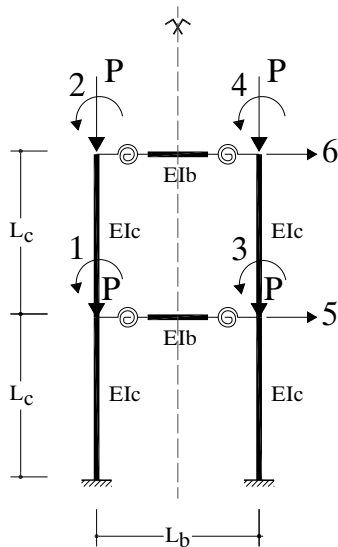


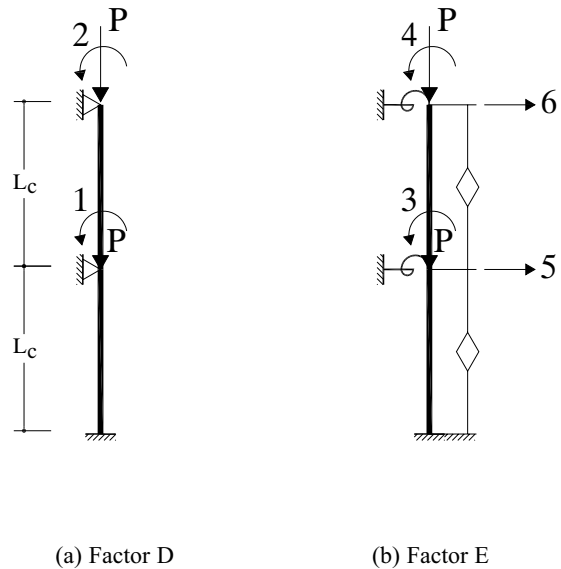
Fig. 8 A two-story frame with two translation DOFs

The stiffness matrix of the frame is constructed as follows:

$$\mathbf{K} = \begin{bmatrix} \frac{8EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & 0 & 0 & -\frac{6EI_C}{L_C^2} \\ \frac{2EI_C}{L_C} & \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & 0 & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{6EI_C}{L_C^2} & -\frac{6EI_C}{L_C^2} \\ \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & 0 & \frac{8EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} & 0 & -\frac{6EI_C}{L_C^2} \\ 0 & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} & \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{6EI_C}{L_C^2} & -\frac{6EI_C}{L_C^2} \\ 0 & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & 0 & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & -\frac{6EI_C}{L_C^2} \\ -\frac{6EI_C}{L_C^2} & -\frac{6EI_C}{L_C^2} & -\frac{6EI_C}{L_C^2} & -\frac{6EI_C}{L_C^2} & -\frac{24EI_C}{L_C^2} & \frac{24EI_C}{L_C^2} \end{bmatrix}$$

$$-\frac{P}{L_C} \begin{bmatrix} \frac{4L_C^2}{15} & -\frac{L_C^2}{30} & 0 & 0 & 0 & -\frac{L_C}{10} \\ -\frac{L_C^2}{30} & \frac{2L_C^2}{15} & 0 & 0 & \frac{L_C}{10} & -\frac{L_C}{10} \\ 0 & 0 & \frac{4L_C^2}{15} & -\frac{L_C^2}{30} & 0 & -\frac{L_C}{10} \\ 0 & 0 & -\frac{L_C^2}{30} & \frac{2L_C^2}{15} & \frac{L_C}{10} & -\frac{L_C}{10} \\ 0 & \frac{L_C}{10} & 0 & \frac{L_C}{10} & \frac{24}{5} & -\frac{12}{5} \\ -\frac{L_C}{10} & -\frac{L_C}{10} & -\frac{L_C}{10} & -\frac{L_C}{10} & -\frac{12}{5} & \frac{12}{5} \end{bmatrix} \quad (17)$$

Having the Form III symmetry, the factors D and E are obtained using the process of healing for the sway frames with odd number of spans, Fig. 9.



(a) Factor D

(b) Factor E

Fig. 9 Factors of the structure shown in Fig. 8

The stiffness matrices of the factors D and E are formed as follows.

$$K_E = \begin{bmatrix} \frac{8EI_C + 12\beta EI_b + 8\gamma EI_b}{L_C + \frac{L_b^2}{L_b}} + \frac{2EI_C}{L_C} & 0 & -\frac{6EI_C}{L_C^2} \\ \frac{2EI_C}{L_C} & \frac{4EI_C + 12\beta EI_b + 8\gamma EI_b}{L_C + \frac{L_b^2}{L_b}} + \frac{6EI_C}{L_C} & -\frac{6EI_C}{L_C^2} \\ 0 & \frac{12EI_C}{L_C^2} & \frac{48EI_C}{L_C^3} - \frac{24EI_C}{L_C^3} \\ -\frac{12EI_C}{L_C^2} & -\frac{12EI_C}{L_C^2} & -\frac{24EI_C}{L_C^3} + \frac{24EI_C}{L_C^3} \end{bmatrix}$$

$$-\frac{P}{L_C} \begin{bmatrix} \frac{4L_C^2}{15} & -\frac{L_C^2}{30} & 0 & -\frac{L_C}{10} \\ \frac{L_C^2}{30} & \frac{2L_C^2}{15} & L_C & -\frac{L_C}{10} \\ 0 & \frac{2L_C^2}{10} & 24 & -\frac{12}{5} \\ -\frac{2L_C^2}{10} & -\frac{2L_C^2}{10} & -\frac{12}{5} & \frac{12}{5} \end{bmatrix}$$

(18a)

$$K_D = \begin{bmatrix} \frac{8EI_C}{L_C} & \frac{2EI_C}{L_C} \\ \frac{2EI_C}{L_C} & \frac{4EI_C}{L_C} \end{bmatrix} - \frac{P}{L_C} \begin{bmatrix} \frac{4L_C^2}{15} & -\frac{L_C^2}{30} \\ -\frac{L_C^2}{30} & \frac{2L_C^2}{15} \end{bmatrix} \quad (18b)$$

Considering  $L_C = 200\text{cm}$ ,  $L_b = 100\text{cm}$ ,  $I = 100\text{cm}^4$ ,  $E = 2.01 \times 10^6 \text{Kg/cm}^2$  and  $k=4 \times 10^6$  for the spring, the buckling loads of the substructures are calculated as:

$$(P_{c\gamma})_D = 37.48 \text{ ton}$$

$$(P_{c\gamma})_E = 27.47 \text{ ton}$$

and the smallest of this is 27.47 ton which is the buckling load of the structure.

#### 4. SYMMETRIC FRAMES WITH AN EVEN NUMBER OF SPANS

In this section frames with an even number of spans are studied. For these frames the axis of symmetry passes through the columns. In general with an appropriate numbering of the

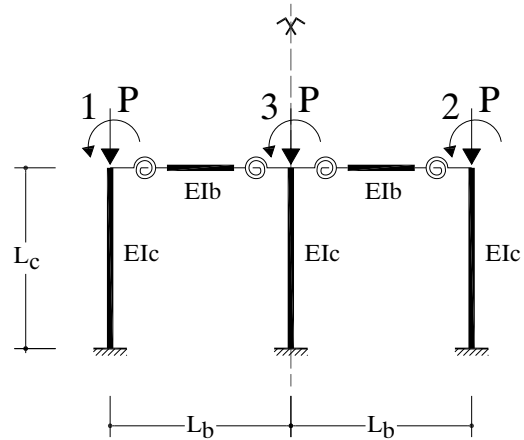


Fig. 10 A two-span frame with no sway

DOFs the stiffness matrices have Form III symmetry. For numbering the rotational DOFs are number suitable for Form II symmetry followed by an arbitrary numbering of the translation DOFs.

#### 4.1 Non-sway Frames

**Example 5:** Consider a two-span non-sway frame with semi-rigid joints as shown in Fig. 10.

The stiffness matrix is constructed as

$$K = \begin{bmatrix} \frac{4EI_C + 6\beta EI_b + 4\gamma EI_b}{L_C + \frac{L_b^2}{L_b}} & 0 & \frac{6\beta EI_b + 4\gamma EI_b}{L_b^2 + L_b} \\ 0 & \frac{4EI_C + 6\beta EI_b + 4\gamma EI_b}{L_C + \frac{L_b^2}{L_b}} + \frac{6\beta EI_b + 4\gamma EI_b}{L_b^2 + L_b} & \frac{6\beta EI_b + 4\gamma EI_b}{L_b^2 + L_b} \\ \frac{6\beta EI_b + 4\gamma EI_b}{L_b^2 + L_b} & \frac{6\beta EI_b + 4\gamma EI_b}{L_b^2 + L_b} & \frac{4EI_C + 12\beta EI_b + 8\gamma EI_b}{L_C + \frac{L_b^2}{L_b}} \end{bmatrix}$$

$$-\frac{P}{L_C} \begin{bmatrix} \frac{2L_C^2}{15} & 0 & 0 \\ 0 & \frac{2L_C^2}{15} & 0 \\ 0 & 0 & \frac{2L_C^2}{15} \end{bmatrix}$$

(19)

Now a new beam element is defined which is



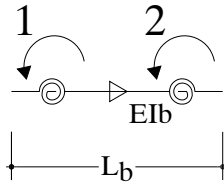


Fig. 11 The new beam element

a directed beam as shown in Fig. 11.

The stiffness matrix for the new beam is as

$$\mathbf{K} = \begin{bmatrix} \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} \\ 0 & 0 \end{bmatrix} \quad (20)$$

The healing process: For frames with an even number of spans the healing process consists of the following steps:

**Step 1:** The structure is cut at an infinitesimal distance  $\varepsilon$  to the left side of the symmetry axis.

**Step 2:** For the substructure at the left hand side, a clamped support is added at the cut points, to obtain the factor D.

**Step 3:** For the substructure at the right hand side, a hinged support are add to prevent the substructure from the lateral displacements. Then directed beams are added at the cut points. In this way the factor E is constructed. For the structure shown in Fig. 10, the factors D and E are shown in Fig. 12.

The stiffness matrices of the factors D and E are constructed as:

$$\mathbf{K}_D = \left[ \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} \right] - \frac{P}{L_C} \left[ \frac{2L_C^2}{15} \right] \quad (21a)$$

$$\mathbf{K}_E = \begin{bmatrix} \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} \\ \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} & \frac{4EI_C}{L_C} + \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} \end{bmatrix} - \frac{P}{L_C} \begin{bmatrix} \frac{2L_C^2}{15} & 0 \\ 0 & \frac{2L_C^2}{15} \end{bmatrix} \quad (21b)$$

Considering  $L_C = 200\text{cm}$ ,  $L_b = 100\text{cm}$ ,  $I = 100\text{cm}^4$ ,  $E = 2.01 \times 10^6 \text{Kg/cm}^2$  and  $k=7 \times 10^6$  for the spring, the buckling loads of the substructures are calculated as:

$$(P_{c\gamma})_E = 150 \text{ ton}$$

$$(P_{c\gamma})_D = 281 \text{ ton}$$

and the smallest of this is 150 ton which is the buckling load of the structure.

**Example 6:** Consider a two-story frame without sway a shown in Fig. 13.

The factors D and E are formed using the process previously described, Fig. 14.

The stiffness matrices of D and E are constructed in the following:

$$\mathbf{K}_D = \begin{bmatrix} \frac{8EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} \\ \frac{2EI_C}{L_C} & \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} \end{bmatrix} - \frac{P}{L_C} \begin{bmatrix} \frac{4L_C^2}{15} & -\frac{L_C^2}{30} \\ -\frac{L_C^2}{30} & \frac{2L_C^2}{15} \end{bmatrix} \quad (22a)$$

$$\mathbf{K}_E = \begin{bmatrix} \frac{8EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & 0 \\ \frac{2EI_C}{L_C} & \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & 0 & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} \\ \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} & 0 & \frac{8EI_C}{L_C} + \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} \\ 0 & \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} & \frac{4EI_C}{L_C} + \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} \end{bmatrix}$$

$$- \frac{P}{L_C} \begin{bmatrix} \frac{4L_C^2}{15} & -\frac{L_C^2}{30} & 0 & 0 \\ -\frac{L_C^2}{30} & \frac{2L_C^2}{15} & 0 & 0 \\ 0 & 0 & \frac{4L_C^2}{15} & -\frac{L_C^2}{30} \\ 0 & 0 & -\frac{L_C^2}{30} & \frac{2L_C^2}{15} \end{bmatrix} \quad (22b)$$

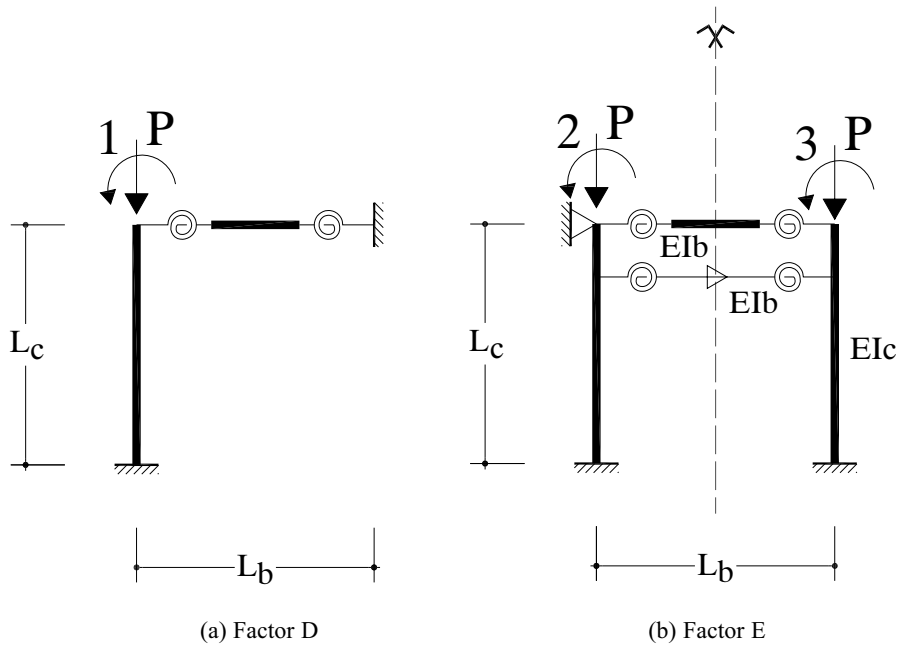


Fig. 12 Factors D and E of the structure shown in Fig. 10

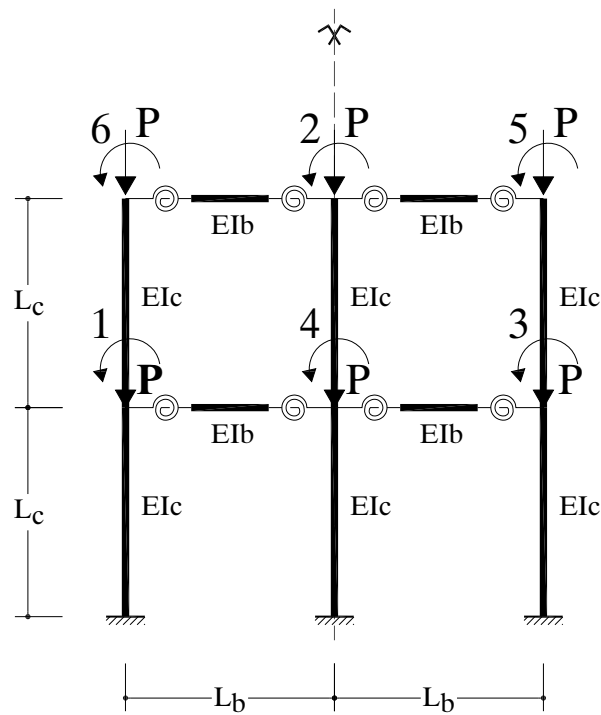


Fig. 13 A two-story frame without sway

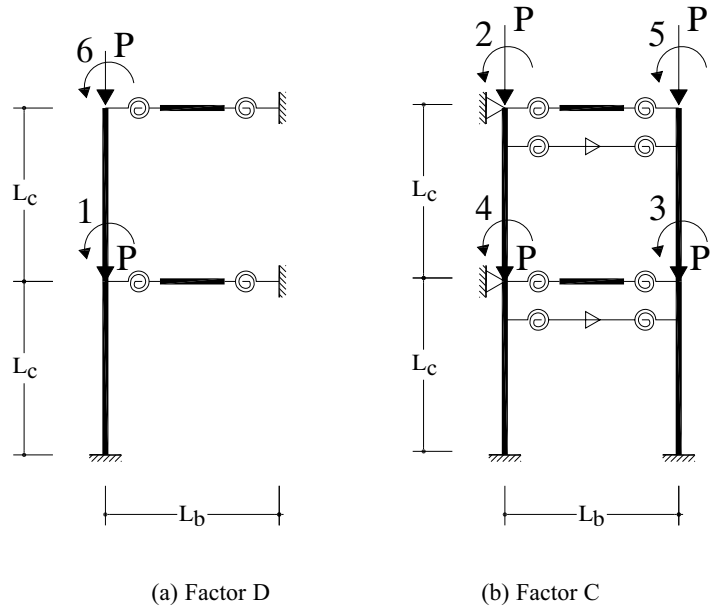


Fig. 14 Factors D and E of the structure shown in Fig. 13

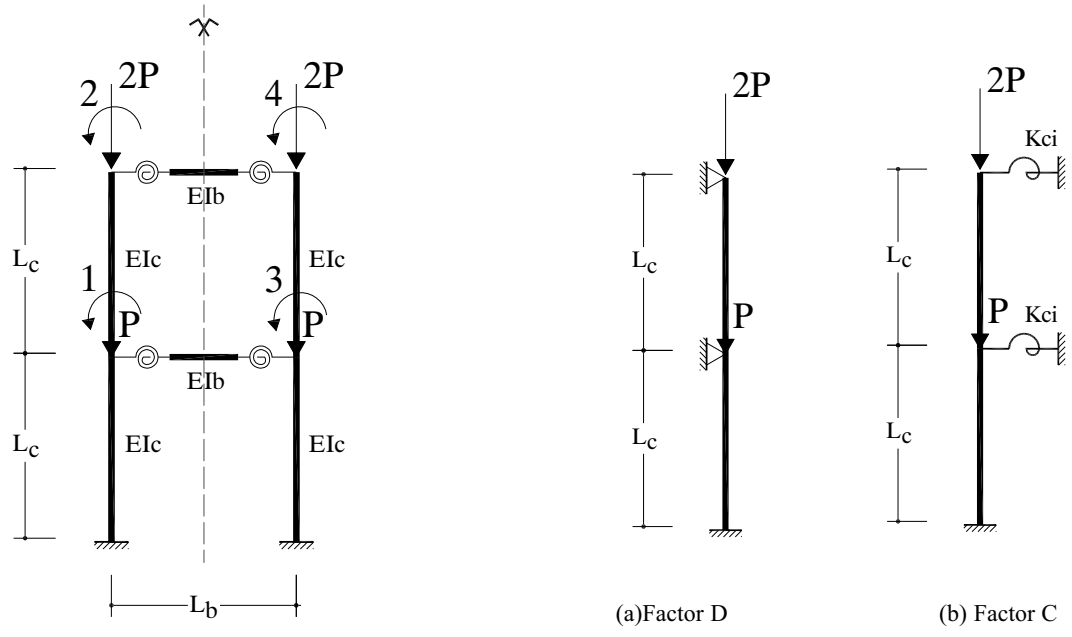


Fig. 15 A symmetric two-story frame

Fig. 16 Factors C and D after healing

Considering  $L_C = 200\text{cm}$ ,  $L_b = 100\text{cm}$ ,  $I = 100\text{cm}^4$ ,  $E = 2.01 \times 10^6 \text{Kg/cm}^2$  and  $k=4 \times 10^6$  for the spring, the buckling loads of the substructures are calculated as:

$$(P_{cr})_D = 95.4 \text{ ton}$$

$$(P_{cr})_E = 37.48 \text{ ton}$$

and the smallest of this is 37.48 ton which is the buckling load of the structure.

**Example 7:** Consider a two-story symmetric frame as shown in Fig. 15.

The stiffness matrix has the Form II symmetry and is formed as:

$$K = \begin{bmatrix} \frac{8EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & 0 \\ \frac{2EI_C}{L_C} & \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & 0 & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} \\ \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & 0 & \frac{8EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} \\ 0 & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} & \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} \end{bmatrix}$$

$$-\frac{P}{L_C} \begin{bmatrix} \frac{6L_C^2}{15} & -\frac{2L_C^2}{30} & 0 & 0 \\ -\frac{2L_C^2}{30} & \frac{4L_C^2}{15} & 0 & 0 \\ 0 & 0 & \frac{6L_C^2}{15} & -\frac{2L_C^2}{30} \\ 0 & 0 & -\frac{2L_C^2}{30} & \frac{4L_C^2}{15} \end{bmatrix} \quad (23)$$

Considering  $L_C = 200\text{cm}$ ,  $L_b = 100\text{cm}$ ,  $I = 100 \text{cm}^4$ ,  $E = 2.01 \times 10^6 \text{Kg/cm}^2$  and  $k = 4 \times 10^6$  for the spring, the buckling load is calculated as

$$(P_{cr}) = 20.94 \text{ ton}$$

Using the suggested healing process the factors C and D are obtained as illustrated in Fig. 16.

The stiffness and mass matrices of the

substructures are constructed as:

$$K_C = \begin{bmatrix} \frac{8EI_C}{L_C} + \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} & \frac{2EI_C}{L_C} \\ \frac{2EI_C}{L_C} & \frac{4EI_C}{L_C} + \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} \end{bmatrix} - \frac{P}{L_C} \begin{bmatrix} \frac{6L_C^2}{15} & -\frac{2L_C^2}{30} \\ -\frac{2L_C^2}{30} & \frac{4L_C^2}{15} \end{bmatrix} \quad (24a)$$

$$K_D = \begin{bmatrix} \frac{8EI_C}{L_C} & \frac{2EI_C}{L_C} \\ \frac{2EI_C}{L_C} & \frac{4EI_C}{L_C} \end{bmatrix} - \frac{P}{L_C} \begin{bmatrix} \frac{6L_C^2}{15} & -\frac{2L_C^2}{30} \\ -\frac{2L_C^2}{30} & \frac{4L_C^2}{15} \end{bmatrix} \quad (24b)$$

Using the same data as in Fig. 15, the buckling load for the substructures are calculated as:

$$(P_{cr})_C = 86.890 \text{ ton}$$

$$(P_{cr})_D = 20.9 \text{ ton}$$

and the smallest of this is 20.9 ton which is the buckling load of the structure.

## 4.2 Sway frames

For this type of frames the corresponding matrices have Form III symmetry when the DOFs are number appropriately as previously described.

**Example 8:** Consider a two-span frame with sway as shown in Fig. 17.

The stiffness matrix has the Form III symmetry and is formed as follows:

$$K = \begin{bmatrix} \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & 0 & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & -\frac{6EI_C}{L_C^2} \\ 0 & \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & -\frac{6EI_C}{L_C^2} \\ \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{4EI_C}{L_C} + \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} & -\frac{6EI_C}{L_C^2} \\ -\frac{6EI_C}{L_C^2} & -\frac{6EI_C}{L_C^2} & -\frac{6EI_C}{L_C^2} & \frac{36EI_C}{L_C^3} \end{bmatrix}$$

$$-\frac{P}{L_C} \begin{bmatrix} \frac{2L_C^2}{15} & 0 & 0 & \frac{L_C}{10} \\ 0 & \frac{2L_C^2}{15} & 0 & -\frac{L_C}{10} \\ 0 & 0 & \frac{2L_C^2}{15} & -\frac{L_C}{10} \\ -\frac{L_C}{10} & -\frac{L_C}{10} & -\frac{L_C}{10} & \frac{18}{5} \end{bmatrix} \quad (25)$$

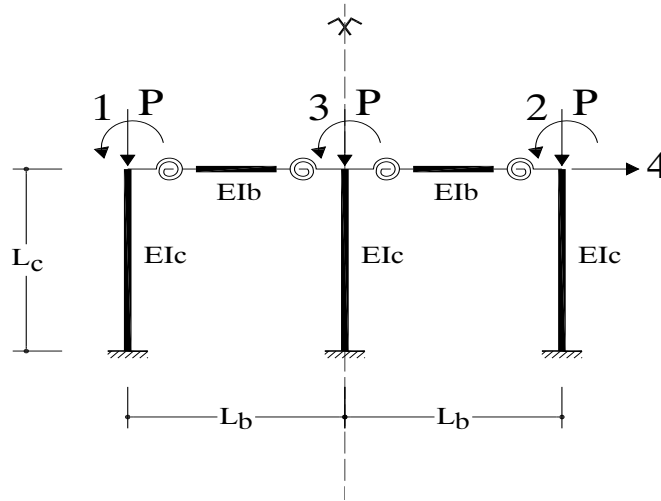


Fig. 17 A symmetric two-span sway frame

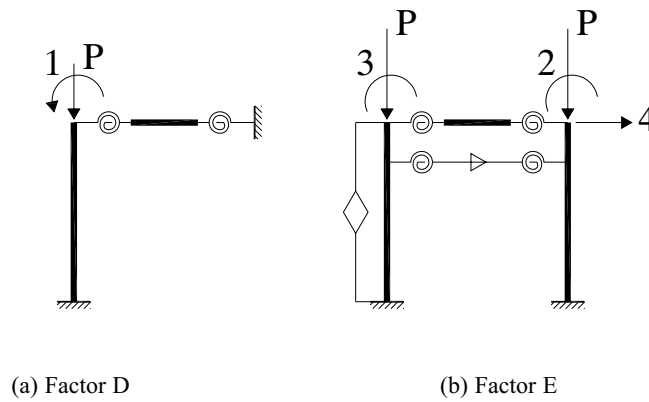


Fig. 18 Factors of the structure shown in Fig. 17

The factors are obtained similar to the previous case, with the difference of a new column being added to the column on the axis of symmetry, Fig. 18. The lateral displacement is incorporated only in the factor E.

The stiffness matrices for the factors D and E are as follow:

$$\mathbf{K}_D = \left[ \begin{array}{c} \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} \\ -\frac{P}{L_C} \left[ \frac{2L_C^2}{15} \right] \end{array} \right] \quad (26a)$$

$$\mathbf{K}_E = \left[ \begin{array}{ccc} \frac{4EI_C}{L_C} + \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & \frac{6\beta EI_b}{L_b^2} + \frac{4\gamma EI_b}{L_b} & -\frac{6EI_C}{L_C^2} \\ \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} & \frac{4EI_C}{L_C} + \frac{12\beta EI_b}{L_b^2} + \frac{8\gamma EI_b}{L_b} & -\frac{6EI_C}{L_C^2} \\ -\frac{12EI_C}{L_C^2} & -\frac{12EI_C}{L_C^2} & \frac{36EI_C}{L_C^3} \end{array} \right] \quad (26b)$$

$$-\frac{P}{L_C} \left[ \begin{array}{ccc} \frac{2L_C^2}{15} & 0 & -\frac{L_C}{10} \\ 0 & \frac{2L_C^2}{15} & -\frac{L_C}{10} \\ -2L_C & -2L_C & \frac{18}{5} \end{array} \right]$$

Considering  $L_C = 200\text{cm}$ ,  $L_b = 100\text{cm}$ ,  $I = 100\text{cm}^4$ ,  $E = 2.01 \times 10^6 \text{Kg/cm}^2$  and  $k=7 \times 10^6$  for the spring, the buckling loads of the substructures are calculated as:

$$(P_{c\gamma})_D = 280 \text{ ton}$$

$$(P_{c\gamma})_E = 38.35 \text{ ton}$$

and the smallest of this is 38.35 ton which is the buckling load of the structure.

## 5. CONCLUDING REMARKS

Exploiting the symmetry of structures can be made by using discrete mathematics. This prepares the ground for more efficient use of the computer and to an understanding which enables us to interpret the final results more readily. Factoring the symmetric structures has the following advantages:

1. The DOFs of the problem is reduced.
2. The computational effort is decreased.
3. The solution of larger problems becomes feasible.

Though the examples are selected from small structures, however, the method shows its potential more when applied to large-scale structures. For structural models with hyper-symmetry, further decomposition can be performed, leading to smaller factors.

Here, only simple types of symmetry are studied corresponding to three canonical forms. The method can be extended to other cases when more than one axis of symmetry is present.

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## Appendix: Analysis of frames with semi-rigid joints

### A.1 End fixity factor

There are two different strategies to incorporate connection flexibility into a computer-base frame analysis. One way is to

define new connection elements to model the beam-column connection directly to the frame analysis. The second approach which is adopted here, is based on modeling the connection as a length-less rotational spring. This method is widely used in semi-rigid analyses, and the implementation of this approach requires small modifications in the existing analysis programs [16].

For the semi-rigid member shown in Fig. A.1, for each connection of the member the following connection fixity can be defined:

$$r_j = \frac{1}{1 + \frac{3EI}{R_j L}} \quad (j=1,2) \quad (\text{A.1})$$

where  $R_j$  is the end-connection spring stiffness and  $EI/L$  is the stiffness of the member. For flexible connections, the value of end fixity factor is zero. For rigid connections, this factor is unity. For a semi-rigid connection this value is between zero and unity.

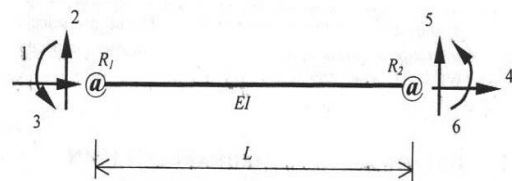


Fig. A.1 A semi-rigid member

### A.2 First Order Analysis

The stiffness matrix of a semi-rigid member  $i$  can be expressed as follows:

$$\mathbf{K}_i^{\text{SR}} = \mathbf{S}_i \cdot \mathbf{C}_i \quad (\text{A.2})$$

where  $\mathbf{S}_i$  is the stiffness matrix of the member with rigid ends and  $\mathbf{C}_i$  is the connection

matrix as:

$$S_i = \begin{bmatrix} \frac{Ea}{L} & 0 & 0 & -\frac{Ea}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \\ & & & \frac{Ea}{L} & 0 & 0 \\ \text{Sym.} & & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ & & & & & \frac{4EI}{L} \end{bmatrix} \quad (A.3)$$

$$C_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4r_2 - 2r_2 + r_1 r_2}{4 - r_1 r_2} & \frac{-2r_1(1 - r_2)}{4 - r_1 r_2} & 0 & 0 & 0 \\ 0 & \frac{6(r_1 - r_2)}{L - (4 - r_1 r_2)} & \frac{-3r_1(2 - r_2)}{4 - r_1 r_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4r_2 - 2r_2 + r_1 r_2}{4 - r_1 r_2} & \frac{-2r_1(1 - r_2)}{4 - r_1 r_2} \\ 0 & 0 & 0 & 0 & \frac{6(r_1 - r_2)}{L - (4 - r_1 r_2)} & \frac{-3r_1(2 - r_2)}{4 - r_1 r_2} \end{bmatrix} \quad (A.4)$$

Where E is Young's modulus, and L, A and I are the length, cross sectional area and moment of inertia, respectively. For  $r_1 = r_2 = r$  we have

$$C_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2r + r^2}{4 - r^2} & \frac{-2Lr(1 - r)}{4 - r^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{3r(2 - r)}{4 - r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2r + r^2}{4 - r^2} & \frac{2Lr(1 - r)}{4 - r^2} \\ 0 & 0 & 0 & 0 & 0 & \frac{3r(2 - r)}{4 - r^2} \end{bmatrix} \quad (A.5)$$

Considering  $\gamma = \frac{3r(2 - r)}{4 - r^2}$ ,  $\beta = \frac{-2r(1 - r)}{4 - r^2}$ ,  $\alpha = \frac{r(r + 2)}{4 - r^2}$

we obtain:

$$C_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & -\beta \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix} \quad (A.6)$$

Hence for the first order analysis we have

$$K_i^{SR} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & \dots & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{\alpha 12EI}{L^3} & \frac{\beta 12EI}{L^3} + \frac{\gamma 6EI}{L^2} & \dots & 0 & -\frac{\alpha 12EI}{L^3} & \frac{\beta 12EI}{L^3} + \frac{\gamma 6EI}{L^2} \\ 0 & \frac{\alpha 6EI}{L^2} & \frac{\beta 6EI}{L^2} + \frac{\gamma 4EI}{L} & \dots & 0 & -\frac{\alpha 6EI}{L^2} & \frac{\beta 6EI}{L^2} + \frac{\gamma 4EI}{L} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{EA}{L} & 0 & 0 & \dots & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{\alpha 12EI}{L^3} & -\frac{\beta 12EI}{L^3} - \frac{\gamma 6EI}{L^2} & \dots & 0 & \frac{\alpha 12EI}{L^3} & -\frac{\beta 12EI}{L^3} - \frac{\gamma 6EI}{L^2} \\ 0 & \frac{\alpha 6EI}{L^2} & \frac{\beta 6EI}{L^2} + \frac{\gamma 2EI}{L} & \dots & 0 & -\frac{\alpha 6EI}{L^2} & \frac{\beta 6EI}{L^2} + \frac{\gamma 4EI}{L} \end{bmatrix} \quad (A.7)$$

### A.3 Second Order Analysis

When an equilibrium and kinematic relationship is established with respect to the deformed geometry of the structure, the analysis is referred to as *geometrical nonlinear* or *second order analysis*.

Consider a planar beam-column as shown in Fig. A.2, consisting of three separate elements with 10 DOFs. For two connections the moment-rotation relationships can be written as

$$\begin{bmatrix} F_7 \\ F_8 \end{bmatrix} = \begin{bmatrix} R_1 & -R_1 \\ -R_2 & R_2 \end{bmatrix} \begin{bmatrix} u_7 \\ u_8 \end{bmatrix}, \quad \begin{bmatrix} F_9 \\ F_{10} \end{bmatrix} = \begin{bmatrix} R_2 & -R_2 \\ -R_2 & R_2 \end{bmatrix} \begin{bmatrix} u_9 \\ u_{10} \end{bmatrix} \quad (A.8)$$

where  $F_7, F_8, F_9$  and  $F_{10}$  are the moments in the connections,  $u_7 - u_8 = \theta_1$  and  $u_9 - u_{10} = \theta_2$  are the rotations of the connections, and  $R_1$  and  $R_2$  are the stiffnesses of connections 1 and 2, respectively.



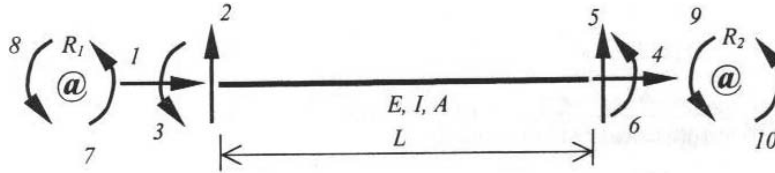


Fig. A.2 Member and end springs with 10 DOFs

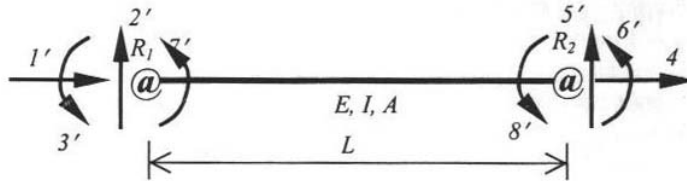


Fig. A.3 A combined element with 8 DOFs

The relationship between member-end displacements and forces for the beam-column considering P-Δ effects is

$$\mathbf{F}_i = (\mathbf{S}_i + \mathbf{G}_i) \cdot \mathbf{u}_i \quad (\text{A.9})$$

Where  $\mathbf{S}_i$  is the elastic stiffness matrix and  $\mathbf{G}_i$  is the geometrical stiffness matrix for a rigid-ended frame and N is the axial force of the member. In an expanded form we have

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} \frac{Ea}{L} & 0 & 0 & -\frac{Ea}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ 0 & 0 & 0 & \frac{Ea}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \text{Sym} & & & & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{6}{5} \frac{L}{10} \\ \frac{2L^2}{15} \\ 0 \\ 0 \\ \frac{6}{5} \frac{L}{10} \end{bmatrix} + \frac{N}{L} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} \quad (\text{A.10})$$

The force-displacement relationship for three elements of Fig. A.2 can be written as:

$$\begin{bmatrix} \mathbf{F}_{6 \times 1} \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \end{bmatrix} = \begin{bmatrix} (\mathbf{S}_i + \mathbf{G}_i)_{6 \times 6} & & & & \\ & R_1 & -R_1 & & \\ & -R_1 & R_1 & & \\ & & & R_2 & -R_2 \\ & & & -R_2 & R_2 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{6 \times 1} \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{bmatrix} \quad (\text{A.11})$$

Or in a compact form we have:

$$\mathbf{F}_{10 \times 1} = \mathbf{F}_{10 \times 10} \cdot \mathbf{U}_{10 \times 1} \quad (\text{A.12})$$

Now consider the three elements of Fig. A.2 to be combined as in Fig. A.3, to form a single element with 8 DOFs.

$$\mathbf{T}_{10 \times 8} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (\text{A.13})$$

The stiffness matrix for the combined element of Fig. A.3 can be written as

$$\mathbf{K}'_{8 \times 8} = \mathbf{T}_{8 \times 10}^t \cdot \mathbf{K}_{10 \times 8} \quad (\text{A.14})$$

$$\mathbf{K}'_{8 \times 8} = \begin{bmatrix} S_{11} + G_{11} & 0 & 0 & S_{14} + G_{14} & 0 & 0 & 0 & 0 \\ 0 & S_{22} + G_{22} & 0 & 0 & S_{25} + G_{25} & 0 & S_{23} + G_{23} & S_{26} + G_{26} \\ 0 & 0 & R_1 & 0 & 0 & 0 & -R_1 & 0 \\ S_{41} + G_{41} & 0 & 0 & S_{44} + G_{44} & 0 & 0 & 0 & 0 \\ 0 & S_{32} + G_{32} & 0 & 0 & S_{55} + G_{55} & 0 & S_{53} + G_{53} & S_{56} + G_{56} \\ 0 & 0 & 0 & 0 & 0 & R_2 & 0 & -R_2 \\ 0 & S_{32} + G_{32} & -R_1 & 0 & S_{35} + G_{35} & 0 & S_{33} + G_{33} + R_1 & S_{63} + G_{63} \\ 0 & S_{62} + G_{62} & 0 & 0 & S_{65} + G_{65} & -R_2 & S_{63} + G_{63} & S_{66} + G_{66} + R_2 \end{bmatrix} \quad (\text{A.15})$$

where  $S_{kl}$  and  $G_{kl}$  for  $k, L = 1, \dots, 6$  are the elements of the elastic and geometrical stiffness matrices, respectively, for the rigid-ended members.

Now the degrees of freedom 7' and 8' are eliminated by static condensation to obtain the force displacement relationship of the element shown in Fig. A.1 with six DOFs.

$$\begin{bmatrix} \mathbf{F}_{6 \times 1} \\ \mathbf{F}_{2 \times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{6 \times 6} & \mathbf{K}_{6 \times 2} \\ \mathbf{K}_{6 \times 2} & \mathbf{K}_{2 \times 2} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{6 \times 1} \\ \mathbf{U}_{2 \times 1} \end{bmatrix} \quad (\text{A.16})$$

Where  $\mathbf{U}_{6 \times 1}$ ,  $\mathbf{F}_{6 \times 1}$  are the displacements and forces corresponding to the exterior DOFs 1' to 6' and  $\mathbf{U}_{2 \times 1}$ ,  $\mathbf{F}_{2 \times 1}$  correspond to DOFs 7' and 8'. From  $\mathbf{F}_{2 \times 1} = \{0, 0\}^t$ , the following can be obtained:

$$\mathbf{K}_{2 \times 6} \cdot \mathbf{U}_{6 \times 1} + \mathbf{K}_{2 \times 2} \cdot \mathbf{U}_{2 \times 1} \quad (\text{A.17a})$$

$$\mathbf{U}_{2 \times 1} = -\mathbf{K}_{2 \times 2}^{-1} \cdot \mathbf{K}_{2 \times 6} \cdot \mathbf{U}_{6 \times 1} \quad (\text{A.17b})$$

leading to

$$\mathbf{F}_{6 \times 1} = (\mathbf{K}_{6 \times 6} - \mathbf{K}_{6 \times 2} \cdot \mathbf{K}_{2 \times 2}^{-1} \cdot \mathbf{K}_{2 \times 6}) \cdot \mathbf{U}_{6 \times 1} \quad (\text{A.18})$$

Therefore

$$\mathbf{K}_i^{\text{SR}} = \mathbf{K}_{6 \times 6} - \mathbf{K}_{6 \times 2} \cdot \mathbf{K}_{2 \times 2}^{-1} \cdot \mathbf{K}_{2 \times 6} \quad (\text{A.19})$$

Each entry of  $\mathbf{K}_i^{\text{SR}}$  involves non-linear fractional functions in terms of the parameter  $NL^2/EA$ , where  $N$  is the member axial force. A Taylor's series is used to expand the non-linear fraction matrix into a polynomial matrix in terms of  $NL^2/EA$ . The elastic stiffness matrix is obtained as the constant matrix of the Taylor's series and the geometrical stiffness matrix  $\mathbf{G}_i$  is obtained as the first order terms of the polynomial matrix and is presented in non-dimensional form as:

$$\mathbf{G}_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & G_{22} & G_{23} & 0 & G_{25} & G_{26} \\ & & G_{33} & 0 & G_{35} & G_{36} \\ & & & 0 & 0 & 0 \\ & & & & G_{55} & G_{56} \\ & & & & & G_{66} \end{bmatrix} \quad (\text{A.20})$$

where

$$G_{22} = \frac{36a}{Dk} \left( k_1^2 k_2^2 + 7k_1^2 k_2 + 7k_1 k_2^2 + 16k_1^2 + 16k_2^2 + 42k_1 k_2 \right)$$

$$G_{23} = \frac{3a}{Dk} (k_1^2 + k_2^2 + 8k_1^2 - 4k_1 k_2^2 - 36k_1 k_2 + 32k_1)$$

$$G_{33} = \frac{4a}{Dk} (k_1^2 k_2^2 + 9k_1^2 k_2 + 24k_1^2)$$

$$G_{36} = \frac{-a}{Dk} (k_1^2 k_2^2 + 12k_1^2 k_2 + 12k_1 k_2^2 + 12k_1 k_2^2 + 84k_1 k_2)$$

$$G_{26} = \frac{3a}{Dk} (k_1^2 k_2^2 + 8k_1 k_2^2 - 4k_1^2 k_2 - 36k_1 k_2 + 32k_2)$$

$$G_{66} = \frac{4a}{Dk} (k_1^2 k_2^2 + 9k_1 k_2^2 + 24k_2^2)$$

$$Dk = (k_1 k_2 + 4k_1 + 4k_2 + 12)^2 \quad (\text{A.21})$$

where

$$G_{55} = -G_{25} = G_{22}, \quad G_{35} = -G_{23}, \quad G_{56} = -G_{26}$$

$$a = \frac{NL^2}{30EI}, \quad K_1 = \frac{R_1 L}{EI}, \quad K_2 = \frac{R_2 L}{EI} \quad (\text{A.22})$$

For simplifying of the analysis, the member geometrical stiffness matrix  $\mathbf{G}_i$  can be expressed in terms of the end-fixity factor  $r_1$  and  $r_2$  can be expressed as follows:

$$\begin{aligned}
G_{22} &= \frac{2N}{5L(4-r_1r_2)^2} (3r_1^2r_2^2 + r_1^2r_2 + r_1r_2^2 + 8r_1^2 + 8r_2^2 - 34r_1r_2 + 40) \\
G_{23} &= \frac{N}{10(4-r_1r_2)^2} (r_1^2r_2^2 + 12r_1^2r_2 + 16r_1r_2^2 - 28r_1r_2^2 + 32r_1^2) \\
G_{33} &= \frac{2NL}{5(4-r_1r_2)^2} (2r_1^2r_2^2 - 7r_1^2r_2 + 8r_1^2) \\
G_{36} &= \frac{-NL}{10(4-r_1r_2)^2} (7r_1^2r_2^2 - 16r_1^2r_2 - 16r_1r_2^2 + 28r_1r_2) \\
G_{26} &= \frac{2N}{10(4-r_1r_2)^2} (r_1^2r_2^2 - 12r_1r_2^2 - 16r_1^2r_2 - 28r_1r_2 + 32r_2^2) \\
G_{66} &= \frac{2NL}{5(4-r_1r_2)^2} (2r_1^2r_2^2 - 7r_1r_2^2 + 8r_2^2) \\
G_{55} &= -G_{25} = G_{22}, G_{35} = -G_{23}, G_{56} = -G_{26}
\end{aligned} \tag{A.23}$$

$$r_1 = \frac{1}{1 + \frac{3EI}{R_1L}}, \quad r_2 = \frac{1}{1 + \frac{3EI}{R_2L}} \tag{A.24}$$

After formation of the geometrical stiffness matrix  $\mathbf{G}_i$ , the overall stiffness matrix of the semi-rigid element  $i$  is calculated as

$$\mathbf{K}_i^{SR} = \mathbf{S}_i \cdot \mathbf{C}_i + \mathbf{G}_i \tag{A.25}$$

Where  $\mathbf{S}_i$  and  $\mathbf{C}_i$  are previously obtained and  $\mathbf{G}_i$  is calculated as (A.20).