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Optimum algorithm for channel flow analysis in direct numerical simulation method

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Abstract

The objective of this work is to perform a direct numerical simulation of turbulent channel flow where all essential scales of motion are resolved due to variable time-stepping algorithm in various time advancement method and different discritized form of convection term. A pseudo spectral method (Fourier series in stream-wise and span-wise directions and Chebychev polynomial expansion in normal direction) is employed for the spatial derivatives. The time advancement is carried out by different semi-implicit and splitting schemes. Also Alternating and Linearized forms are added to four commonly used forms of the convective term, referred to as divergence, Convection, skew-symmetric, and rotational. Spectral method based on the primitive variable formulation is used in Cartesian coordinates with two periodic and one non-periodic boundary condition in three dimensional directions $\Omega=[0,4\pi]\times[-1,1]\times[0,2\pi]$. The friction Reynolds number for channel flow is set to be $Re_\tau=175$ and the computational grids of $128\times 65\times 128$ are used in the x, y and z directions, respectively. The comparison is made between turbulent quantities such as the turbulent statistics, wall shear velocity, standard deviation of u and total normalized energy of instantaneous velocities in different time-discretization methods and different forms of nonlinear term. The present results show that third-order time-discretizations forward Euler for explicit terms and backward Euler for implicit terms can minimize the computational cost of integration by maximizing the time step, while keeping the CFL number near a threshold in time-discretization schemes. Also, the de-aliased results of turbulence statistics do indicate that different expressions of nonlinear terms have minor discrepancy in pseudo spectral method. The results show that the most desirable approach is a combination of variable time stepping third order backward difference algorithm and rotational form, which provides reduced cost and further accuracy improvements

Keywords: Channel flow, Pseudo spectral method, DNS, time-discretization schemes, nonlinear term forms.

1. Introduction

Practically most of the fluid flows either generated by nature (e.g., oceans, winds, rivers) or by human industrial activity (e.g., planes, cars, materials processing, biomedical engineering) are turbulent ones. There are different common ways to obtain the information about a flow field with numerical simulations like [1] and [2]. These approaches have many constraints, especially for turbulent flows. There is a need to model turbulent fluid flows in order to improve the basic understanding of these complex phenomena and to increase the design quality of technological applications. The non-linearity in the Navier-Stokes equations gives rise to a wide range of spatial and temporal turbulent scales. However, a complete description of a turbulent flow, where the flow variables (e.g. velocity and pressure) are known as a function of space and time with the resolution of all scales can only be obtained by numerically solving the Navier-Stokes equations [3]. These numerical solutions are called direct numerical simulations (DNS). Direct numerical simulation is a time-dependent and three-dimensional numerical solution in which the governing equations are computed as accurately as possible without using any turbulence models. It provides a wide range of information such as velocity, pressure and their derivatives at any time and space in the flow field. These are extremely difficult to be measured in experiments [4].

Still with such features, the method demands higher computer memory compared to the conventional techniques. In recent years, the developments of supercomputers facilitate such numerical simulations of fluid flows in different cases [5]. However, with present resources, a direct simulation is

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feasible for simulation of transient, three-dimensional (3D) flows in or around simple geometries at low and moderate Reynolds numbers. Nowadays both finite-difference schemes and spectral schemes are used for direct numerical simulations of the fluid flows .Yet it seems that spectral methods are the best tools to achieve higher accuracy on a simple domain and demand less computer memory than other alternatives [3].

The first attempt of the DNS was made by Orszag and Patterson (1972) to perform computation of isotropic turbulence at a Reynolds number (based on Taylor micro scale) of 35 [6]. More recently, the DNS of the fully developed turbulent channel flow started for the wall turbulence. Actually the plane channel DNS was preceded by DNS of the flow in a curved channel. It was not until 1987 that DNS of the plane channel was performed by Kim et al. (1987) [7]. Their Reynolds number based on the friction velocity u_t and the channel half width d was Re_τ=180. Since then, the channel flow has proven to be an extremely useful framework for the study of wall-bounded turbulence. It has often been performed because of its simple geometry and fundamental nature to understand the transport mechanism. Afterwards various Reynolds number channel simulations have been performed by many others. Kuroda et al. [8] and Kasagi et al. [9] carried out the DNS for a slightly lower Reynolds number of Re_τ=150. Kim et al. [10] also performed a DNS with a higher Reynolds number of Re_τ=395. Kawamura et al. [11, 12] performed the DNS to include the scalar transport with various Prandtl numbers for Re_τ=180 and 395. They also carried out the DNS for a higher Reynolds number of Re_τ=640 and reported preliminary results in Kawamura [13] and Kawamura et al. [14]. In recent years, numerical simulations of fully developed turbulent channel flow at Reynolds numbers up to Re_τ=2320 [15] and Re_τ=2000 [16] are also reported.

There is a rich variety of strategies for time discretizing of the Navier–Stokes equations in direct numerical simulations. Most commonly used time-discretization strategies are splitting techniques and coupled methods (monolithic methods). The progenitor of splitting methods in fluid dynamics is the Chorin–Temam method proposed by Chorin (1968) and Temam (1969) in the late 1960s. In the late 1970s, second-order Adams–Bashforth for explicit terms and second-order Crank–Nicolson for implicit terms were common choices [17]. Low-storage Runge–Kutta (third-order and fourth-order) for explicit terms became popular in the 1980s [18]. In the 1990s, the third-order backward difference scheme came into use for implicit terms. Potentially, more accurate splitting scheme is existed based on a higher order discretization of the time derivative [17].

This paper presents results from Direct Numerical Simulations (DNS) of fully developed plane channel flow. This is a prototypical flow frequently used to study physical and numerical modeling of wall-bounded flows. The mathematics of numerical method are based on the pseudo spectral algorithm, proposed by Canuto et al. [19], with spectral discretization in spatial directions (Fourier × Chebyshev ×Fourier) and finite-differencing in time. The differential equations are Helmholtz equations. The tau-equation solution with tau correction is also used in their discretized form for each Fourier mode. Primitive variables (3d velocity and pressure) are used to integrate the incompressible Navier-Stokes equations. The time advancement is carried out by both semi-implicit schemes and splitting methods. Here, we report two-stage scheme based on a BDF (backward difference algorithm) treatment of the linear terms, combined with an explicit extrapolation of the nonlinear convection terms.

The primitive variable form of the three-dimensional incompressible Navier-Stokes equations has several equivalent versions due to the precise manner of expressing the nonlinear terms. The more common alternatives are the convection form, the divergence form, the skew-symmetric form and the rotation form [20]. Zang found that the collocation results based on the rotation form were decidedly inferior to those based on the skew-symmetric (or the more economical alternating) form, but de-aliased rotation form performed quite admirably. Nevertheless, there isn't a comprehensive report of their usage in de-aliased Poiseuille channel flow in x and z directions. In this work, our first concern is to reduce the computational cost, so a new variable time stepping method is applied by adding an accessory (supplementary) algorithm. This algorithm is used to minimize the computational cost of integration by maximizing the time step, while the CFL number keeps near a threshold. The CFL number and time step are bound in a given range to control the stability. This time step determines as a fraction of a fixed time-interval to keep CFL number maximum under above condition. The results of direct numerical simulations for turbulent channel flow with six different forms of nonlinear terms and six different time-discretization methods in variable time stepping manner are compared to choose the most economic, accurate and stable algorithm.

2. Numerical Method

The channel flow is a remarkable example of wall bounded problems. In this study, a plane Poiseuille flow with parabolic stream-wise velocity profile and no slip occurring at the planes is considered. The flow geometry and the coordinate system are shown in figure 1.

Fully developed turbulent channel flow is homogeneous on a rectangular, wall bounded domain. Flow fields are allocated in terms of their physical grid sizes $N_x \times N_y \times N_z$ on the computational domain $\Omega=[0,L_x]\times[a,b]\times[0,L_z]$. L_x and L_z are respectively the periods in two infinite stream wise and span wise directions and b-a is the height of the channel. In this case

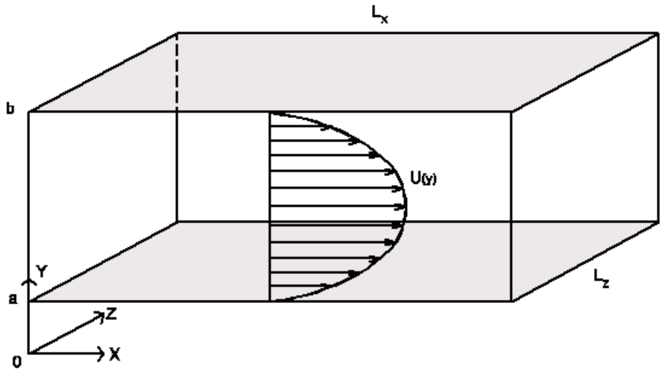

Fig. 1. Schematic of channel flow

the flow domain is set to be 2 units high, 2π units wide and 4π units long that periodically continued in span-wise and stream-wise direction. The flow is integrated on a 128×65×128 grid points. No-slip boundary conditions at y=∓1 are assumed. The initial Reynolds number is 4000 so the viscosity set to be v=1/4000. The velocity flow represents by vector-valued Fourier × Chebyshev × Fourier expansions whose mathematical form is as follows;

$$\mathrm{u(x)}=\sum_{k_x=-N_x/2+1}^{N_x/2}\sum_{n_y=0}^{N_y-1}\sum_{k_z=-N_z/2+1}^{N_z/2}\hat{\tilde{\mathrm{u}}}_{kx,ny,kz}\bar{T}_m(y)e^{2\pi i(k_x x/L_x+k_z z/L_z)} \tag{1}$$

where, X=(x,y,z). The double tilde/hat notation on the spectral coefficients $\hat{\tilde{\mathrm{u}}}_{kx,ny,kz}$ indicates that the coefficients result from a combined Fourier transform in xz and a Chebyshev transform in y. Here $\bar{T}_m$ is the m^{th} Chebyshev polynomial rescaled to the interval y∈[a,b]. That is,

$$\bar{T}_m=T_m\left(\frac{2y-(b+a)}{b-a}\right)$$

and T_m is Chebyshev polynomial of the usual domain [-1,1]. The spectral coefficients of u can be computed efficiently from the function values taken at a discrete set of Chebyshev grid points in the form of;

$$y_n=\frac{b+a}{2}+\frac{b-a}{2}\cos\left(\frac{n\pi}{N_y-1}\right)\qquad n\in(0,N_y-1) \tag{2}$$

The discretization in the horizontal directions are done using Fourier series expansions thus assuming periodicity, which is reasonable if the flow is homogeneous in these directions. Here Nx and Nz are the number of Fourier modes included in the respective directions.

3. Governing Equations

The Navier-Stokes equations for an incompressible flow in the channel flow geometry can be written in the following form:

$$\frac{\partial \mathrm{u}_{tot}}{\partial t}+\mathrm{u}_{tot}\cdot\nabla\mathrm{u}_{tot}=-\nabla p_{tot}+\nu\nabla^2\mathrm{u}_{tot} \tag{3}$$

$$\nabla u_{tot}=0 \tag{4}$$

Where $u_{tot}(x;t)$ is the total fluid velocity field with u,v and w components in three dimensions and $p_{tot}(x;t)$ is the total pressure field. The first and second equations represent conservation of momentum and incompressibility of the fluid, respectively. The velocity satisfies the no-slip boundary conditions ($u=0$) at both upper and lower channel walls ($y=a,b$). The boundary conditions in the x and z directions are periodic: $u_{tot}(x+L_x,y,z;t)=u_{tot}(x,y,z;t)$ and $u_{tot}(x,y,z+L;t)=u_{tot}(x,y,z;t)$

Total velocity and pressure fields can be broken into constant and fluctuating parts, so the velocity field is the sum of the base velocity or base flow $U(y)e_x$, and the fluctuating velocity $u(x;t)$. The total pressure field is also the sum of a linear-in-x term $\Pi_x(t)\,x$ and a periodic fluctuating pressure $p(x;t)$. Also the gradient of this decomposition relates the total pressure gradient to a spatially-constant base pressure gradient $\Pi_x e_x$ and a fluctuating pressure gradient $p(x;t)$. Therefore;

$$u_{tot}(x;t)=\mathrm{U(y)}e_x+u(x;t) \tag{5}$$

$$p_{tot}(\mathrm{x};t)=x\frac{dP}{dx}(t)+p(\mathrm{x};t)=\prod_x(t)x+p(\mathrm{x};t) \tag{6}$$

$$\nabla p_{tot}(\mathrm{x};t)=\frac{dP}{dx}(t)e_x+\nabla p(\mathrm{x};t)=\prod_x(t)e_x+\nabla p(\mathrm{x};t) \tag{7}$$

Substituting equations 5 and 7 into equation 3 gives:

$$\frac{\partial u}{\partial t}+\nabla p=\nu\nabla^2\mathrm{u}-\mathrm{u}_{tot}\cdot\nabla\mathrm{u}_{tot}+\left[\nu\frac{\partial^2 U}{\partial y^2}-\Pi_x\right]e_x \tag{8}$$

There are several different forms for the nonlinear term $u_{tot}\cdot\nabla u_{tot}$in equation 8 that are identical in continuous mathematics but have different properties when discretized.

The convection form $u_{tot}\,.\,\nabla u_{tot}$ (9)
The divergence form $\nabla.(u_{tot}\ u_{tot})$ (10)
The skew-symmetric form $1/2\,u_{tot}.\nabla u_{tot}+1/2\nabla.(u_{tot}u_{tot})$ (11)
The rotational form $(\nabla\times u_{tot})\times u_{tot}+1/2\,\nabla.(u_{tot}\ u_{tot})$ (12)

In this paper, the nonlinear term is first expanded with the base-fluctuation decomposition and then, calculates the nonlinear term from these forms:

$$\frac{\partial \mathrm{u}}{\partial t}+\nabla q=\nu\nabla^2\mathrm{u}-\mathrm{N(u)}+\left[\nu\frac{\partial^2 U}{\partial y^2}-\Pi_x\right] \tag{13}$$

$$q\quad\begin{cases}p+1/2\,\mathrm{u.u} & \text{Rotational}\\ p & \text{else}\end{cases} \tag{14}$$

The forms of nonlinear term, defined by N (u), which could substitute in this code, are as follows;

$$\mathrm{N(u)}\quad\begin{cases}(\nabla\times\mathrm{u})\times\mathrm{u}+U\dfrac{\partial\mathrm{u}}{\partial x}+\nu\dfrac{\partial U}{\partial y}\mathrm{e}_x & \text{Rotational}\\ \mathrm{u}_{tot}\cdot\nabla\mathrm{u}_{tot} & \text{Convection}\\ \nabla\cdot(\mathrm{u}_{tot}\,\mathrm{u}_{tot}) & \text{Divergence}\\ \dfrac{1}{2}\mathrm{u}_{tot}\cdot\nabla\mathrm{u}_{tot}+\dfrac{1}{2}\nabla\cdot(\mathrm{u}_{tot}\mathrm{u}_{tot}) & \text{Skew-symmetric}\\ U\dfrac{\partial\mathrm{u}}{\partial x}+\nu\dfrac{\partial U}{\partial y}\mathrm{e}_x & \text{Linearized}\\ \text{Alternating form (: equation 10 and 11 on alternating time steps)}\end{cases} \tag{15}$$

The details and mathematics are based on the spectral channel flow algorithm proposed by Cantou et al. [19]. Equation 13 is solved by the Chebychev-tau method for each wave number after it is Fourier transformed in the stream wise and spans wise directions. Also, there is a need to add tau correction to the solution of the equations in their discretized form which is used to determine the pressure.

4. Time Advancement

The time advancement is carried out by 6 time-integration schemes. All schemes treat the linear term implicitly and the nonlinear term explicitly. The initial time step is set to a definite dt_0 or could be determined by the average of minimum and maximum bounds and may vary during the integration $dt=dt_0/n$.

In formulation of time integration scheme, linear term L (u) and the constant term C in Navier-Stokes equation are defined by:

$$\mathrm{L\,u} \triangleq \nu \nabla^2 \mathrm{u} \tag{16}$$

$$\mathrm{C} \quad \left[\nu \frac{\partial^2 U}{\partial y^2} - \Pi_x \right] \mathrm{e}_x \tag{17}$$

With these definition the equation (13) can be written as:

$$\frac{\partial \mathrm{u}}{\partial t} + \nabla q = \mathrm{Lu} - \mathrm{N(u)} + \mathrm{C} \tag{18}$$

After Fourier transform, equation 15 can be written as follows;

$$\frac{\partial \tilde{\mathrm{u}}}{\partial t} + \tilde{\nabla} \tilde{q} = \tilde{\mathrm{L}} \tilde{\mathrm{u}} - \mathrm{N(u)} + \tilde{\mathrm{C}} \tag{19}$$

Let $\tilde{\mathrm{u}}^{n}$ be the approximation of $\tilde{\mathrm{u}}$ at time $t=n\Delta t$ and $\tilde{N}^n \triangleq \widetilde{N(u^n)}$ .Here, we use three time advancement methods as examples for coupled (monolithic) techniques .They are mixed Crank-Nicolson/ second order Adams-Bashforth scheme (CNAB2), 3rd order Runge-Kutta scheme combined to Crank-Nicolson method (CNRK3) and joined second order Spalart-Moser, Runge-Kutta scheme (SMRK2). To determine the terms in equation 19 at $t=(n-1/2)\Delta t$ they would approximate as follows

$$\frac{\partial}{\partial t} \tilde{\mathrm{u}}^{n+1/2} = \frac{\tilde{\mathrm{u}}^{n+1} - \tilde{\mathrm{u}}^{n}}{\Delta t} + \mathrm{O}\left(\Delta t^2\right) \tag{20}$$

$$\tilde{\mathrm{L}}\tilde{\mathrm{u}}^{n+1/2} = \frac{1}{2}\tilde{\mathrm{L}}\tilde{\mathrm{u}}^{n+1} + \frac{1}{2}\tilde{\mathrm{L}}\tilde{\mathrm{u}}^{n} + \mathrm{O}\left(\Delta t^2\right) \tag{21}$$

$$\tilde{\nabla}\tilde{q}^{n+1/2} = \frac{1}{2}\tilde{\nabla}\tilde{q}^{n+1} + \frac{1}{2}\tilde{\nabla}\tilde{q}^{n} + \mathrm{O}\left(\Delta t^2\right) \tag{22}$$

$$\tilde{\mathrm{N}}^{n+1/2} = \frac{3}{2}\tilde{\mathrm{N}}^{n} - \frac{1}{2}\tilde{\mathrm{N}}^{n-1} + \mathrm{O}\left(\Delta t^2\right) \tag{23}$$

$$\tilde{\mathrm{C}}^{n+1/2} = \frac{1}{2}\tilde{\mathrm{C}}^{n+1} + \frac{1}{2}\tilde{\mathrm{C}}^{n} + \mathrm{O}\left(\Delta t^2\right) \tag{24}$$

The time-derivative approximations for the linear terms are Crank-Nicolson, and that of the nonlinear term (N) is second order Adams-Bashforth, which could plug into equation 15 for different time-integration scheme. Different coefficients α_i, β_i, γ_i and ζ_i could use for each sub-step(Δt) in a time step and rearranged as follows;

$$\left[\frac{1}{\Delta t} - \beta_i \tilde{\mathrm{L}}\right] \tilde{\mathrm{u}}^{n,i+1} + \beta_i \tilde{\nabla}\tilde{q}^{n,i+1} =$$

$$\left[\frac{1}{\Delta t} - \alpha_i \tilde{\mathrm{L}}\right] \tilde{\mathrm{u}}^{n,i} - \alpha_i \tilde{\nabla}\tilde{q}^{n} + \gamma_i \tilde{\mathrm{N}}^{n,i} + \zeta_i \tilde{\mathrm{N}}^{n,i-1} + \beta_i \tilde{\mathrm{C}}^{n+1} + \alpha_i \tilde{\mathrm{C}}^{n} \tag{25}$$

The second superscript i indicates the time method sub-steps [21]. 3rd order Runge-Kutta scheme is a low-storage Runge-Kutta method, which is proposed to minimize storage demands in spectral methods. The general representation with 2 levels of storage is as follows;

$$u_0 = u^n$$
$$Q_j = A_j Q_{j-1} + \Delta t\, N(u_{j-1})$$
$$u_j = u_{j-1} + B_j Q_{j-1} + B_j \Delta t[L(u_{j-1}) + L(u_j)] \qquad j=1,\dots, s=3$$
$$u^{n+1} = u_3 \tag{26}$$

Note that A_i, B_i, B'_i are same coefficients as α_i, β_i, γ_i based on semi implicit three stage Runge-Kutta algorithm in "Spectral methods for incompressible viscous flow" by Roger Peyret [22].

Splitting method is another class of implicit time discretizations which is used in our research. This scheme leads to a pressure-correction algorithm based on a backward difference algorithm (BDF) treatment of the linear terms combined with an explicit extrapolation of the nonlinear convection terms. Higher order discretization of the time derivative based on backwards difference formulas is used for linear terms. Here, a three-stage scheme based on a BDF treatment of the linear terms combined with an explicit extrapolation of the nonlinear convection terms is applied. Let $J \geq 1$ be the number of steps and $\zeta_0, \alpha_0, \alpha_1, \dots, \alpha_{j-1}$ the coefficients, of the BDF formula for the discretization, therefore the time derivative d_y/d_t at time $n-1$ is

$$\left.\frac{dy}{dt}\right|_{t=t^{n+1}} \quad \frac{1}{\Delta t}\left(\zeta_0 y^{n+1} - \sum_{q=0}^{J-1} \alpha_q y^{n-q}\right) \tag{27}$$

Moreover, let $\beta_0, \beta_1, \dots, \beta_{j-1}$ be the coefficients of the extrapolation formula for nonlinear term [21], thus we have;

$$N^{n+1} = \sum_{q=0}^{J-1} \beta_q \mathrm{N}^{n-q} \tag{28}$$

The first stage consists of solving the explicit problem in the forms of;

$$\frac{1}{\Delta t}\left(\hat{\mathrm{u}}^{n+1} - \sum_{q=0}^{J-1} \alpha_q \mathrm{u}^{n-q}\right) = \sum_{q=0}^{J-1} \beta_q \mathrm{N}^{n-q} \quad \text{in } \Omega \tag{29}$$

The second stage is the projection step as follows;

$$\frac{1}{\Delta t}\left(\hat{\hat{\mathrm{u}}}^{n+1} - \hat{\mathrm{u}}^{n+1}\right) + \nabla p^{n+1} = 0 \text{ in } \Omega$$
$$\nabla . \hat{\hat{\mathrm{u}}}^{n+1} = 0 \text{ in } \Omega$$
$$\hat{\hat{\mathrm{u}}}^{n+1} . n = 0 \quad \text{on } \partial\Omega_0 \tag{30}$$

And the third stage is the diffusion step in the following form;

$$\frac{1}{\Delta t}\left(\zeta_0 \mathrm{u}^{n+1} - \hat{\hat{\mathrm{u}}}^{n+1}\right) - \nu\, \Delta u^{n+1} = 0 \text{ in } \Omega$$

$$u^{n+1} = 0 \quad \text{on } \delta\Omega_0 \tag{31}$$

Time-integration schemes and coefficient values for each time method are reviewed in table 1.

5. Results and discussion

A direct numerical simulation of turbulent channel flow is carried out on a periodic, rectangular, wall bounded domain with128 × 65 × 128 mesh points at the friction Reynolds number of $Re_\tau = 175$ based on a friction velocity and channel half width and initial viscosity of $\nu = 1/4000$. A fully pseudo spectral method of Fourier series in the homogeneous directions and Chebyshev polynomial expansion in the normal direction are used for the spatial derivatives with different time advancement schemes. De-aliased XZ method is employed to vanish aliasing error at x and z (not y) directions and different forms of nonlinear term $u_{tot} \,.\, \nabla u_{tot}$ are used to illustrate the effect of various algorithms. The results of turbulence statistics have been collected and compared with different forms of time advancements and nonlinear terms. The comparison of

Table 1. Time-integration schemes and their Characteristics (coefficient)

Scheme	Non linear Term	Linear Term	A	β	γ	ζ
CNAB2	2nd order Adams-Bashforth	Crank-Nicolson	α_0= 0.5	β_0=0.5	γ_0=1.5	ζ_0= -0.5
CNRK3	3rd order Runge-Kutta	Crank-Nicolson	α_0=0 , α_1= -5/9 α_2=-153/128	β_0=1/3, β_1=15/16, β_2= 8/15	γ_0= 1/6 , γ_1=5/24, γ_2= 1/8	-
SMRK2	Runge-Kutta	Spalart_Moser	α_0= 29/96 , α_1=-3/40 α_2=1/6	β_0= 37/160 , β_1=5/24 β_2=1/6	γ_0= 8/15, γ_1= 5/12 γ_2=3/4	ζ_0=0 , ζ_1= -17/60, ζ_2=-5/12
SBDF2	2nd order BDF	2nd order extrapolation	α_0=-2, α_1=0.5	β_0=2 , β_1=-1	-	ζ=1.5
SBDF3	3rd order BDF	3rd order extrapolation	α_0=-3 ,α_1= 1.5 α_2=-1/3	β_0= 3, β_1= -3 β_2=1	-	ζ=11/6
SBDF4	4th order BDF	4th order extrapolation	α_0= -4 ,α_1=3 α_2=-4/3 ,α_3=.25	β_0=4, β_1=-6 β_2=4 , β_3=-1	-	ζ=25/12

turbulence intensities is made over the interval from T_0=100 to T_1=300 due to [4] and the time step of recorded information was dT=10 sec. The initial time step dt_0 set to .02 and the variable time step is bounded from dt_{min}=0.002 to dt_{max}=0.04. Also the CFL condition is adjusted to [0.4,0.8].

Figure 2 shows the results of stream-wise turbulence strength value in different time advancement schemes, using the rotational form to extract the nonlinear term. Despite the qualitative agreement in all forms, detailed comparison in the middle of the channel reveals discrepancies between SBDF2 and the other schemes. In this region, the results of SBDF2 as higher values than the other forms. The average values of stream-wise turbulence strength (uu) at each grid point of normal direction in all time advancements are computed, except for SBDF2 scheme and the results are compared. The deviation of averaged values is 1.94E-03 (156%) in SBDF2 scheme, while the obtained maximum deviation in other scheme is about 4.00 E-4 (3.36%) at SBDF4 method. Also the trend of normal and span wise turbulence strength profile of SBDF2 scheme is completely different from Kim et al. result [7] in the middle of the channel height.

Figure 3 & 4 show that comparison of turbulence strength that revealed consistence discrepancy in the middle of the channel. The maximum deviation of averaged vv values obtained from CNAB2 method is about 6.02E-05 (8.69%) and 3.33E-03 (482%) from SBDF2 scheme. This maximum deviation of averaged ww is reached to 1.21E-04 (5.28%) in SBDF4 scheme but is about 3.54E-03 (490%) in SBDF2 scheme.

Table 2 reviews the values of wall-shear velocity, standard deviation, overall CPU time, total normalized energy of instantaneous velocities(u), dissipation and normalized energy of U(base flow velocity) and u(instantaneous stream wise velocity) . The computed values are providing with six time

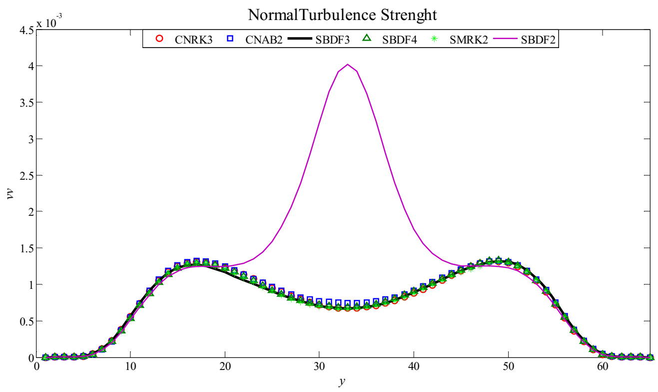

Fig. 3. Normal turbulence strength with different time advancement scheme

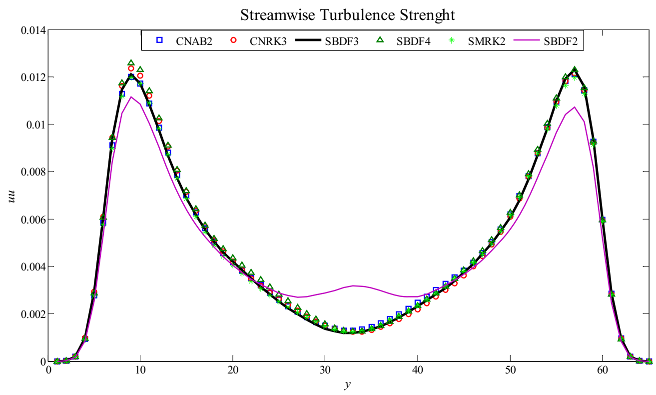

Fig. 2. stream-wise turbulence strength with different time advancement scheme

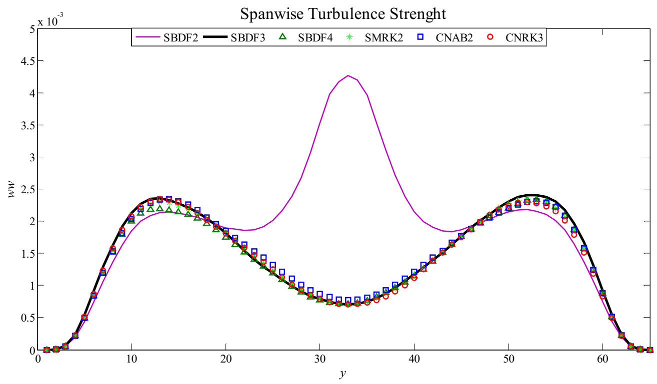

Fig. 4. Span-wise turbulence strength with different time advancement scheme

advancement schemes. The standard deviation of u in each method is computed by;

$$\sqrt{\frac{1}{N}\sum_{i=T_0}^{T_1}\frac{1}{L_x L_y L_z}\int_0^{L_x}\int_a^{b}\int_0^{L_z}(\mathrm{u_i}-\mathrm{u}_{\mathrm{i}mean})\cdot(\mathrm{u_i}-\mathrm{u}_{\mathrm{i}mean})dx\,dy\,dz} \quad (32)$$

And the total normalized energy of instantaneous velocities is calculated by:

$$\sum_{i=T_0}^{T_1}\frac{1}{L_x L_y L_z}\int_0^{L_x}\int_a^{b}\int_0^{L_z}\mathrm{u_i}\cdot\mathrm{u_i}\,dx\,dy\,dz \quad (33)$$

The differences between the values of SBDF2 and other schemes are noticeable. In all versions, the calculated values of turbulent specifications are almost similar except for the SBDF2 scheme. For example, according to the table 2, dissipation values of SBDF2 scheme differs about 1.75% (.08835) of the average values from other schemes whereas the maximum deviation from average values of other 5 schemes is 17% (.86).

The variation of CFL number during the operation with SBDF2 is shown in Figure 5. The figure shows sharp fluctuating which may make the solution unstable. In practical computations, one uses schemes that yield higher accuracy under milder stability restrictions. Both higher-order time discretization and smaller time step could be applied to achieve the desirable solution and avoid weak instability. (The notion of weak instability is used in a loose sense for schemes which admit solutions to periodic hyperbolic problems that grow with time, but for which the growth rate decreases with Δt [21]). It looks that we have weak instability in lower-order time discretization like SBDF2 scheme. For such weakly instability, in the long the time interval of interest, the Δt must be chosen smaller to keep the spurious growth of the solution within acceptable bounds. We propose as the modified SBDF2 to eliminate the source of instability and discrepancy is by decreasing the initial time step (*dt*) from 0.02 to 0.01. Also, the minimum bound of time step (dt_{min}) should be decreased from 0.002 to 0.0005 and the CFL condition had to be adjusted by changing the given interval of CFL number from [0.4,0.8] to [0.2,0.6]. These changes increase the overall CPU time up to 4.1111e+04 and decrease the dominant CFL condition from about 0.5 to 0.25. As a result, the values of wall shear velocity, energy (u) and dissipation of *U+u* change to 0.04380108, 0.0191442 and 5.13975 in modified SBDF2, which are very close to the values of other time advancement schemes.

Figure 6 compares the result of modified SBDF2 and other time advancement methods. According to the results, the difference between the results of uu turbulence strength from modified SBDF2 and average values from other 5 schemes is about 2.63% (2.08E-04). The resulted differences are 1.72% (2.03E-05) for *vv* and 1.69% (3.96E-05) for *ww*. The maximum deviation from average values is 3.13% (3.73E-04) for *uu* at SBDF4, 8.74% (6.06E-05) for *vv* at CNAB2 and 4.97% (9.29E-05) for *ww* at SBDF3 scheme. The deviation of averaged values of turbulence statistics between SBDF2 and other schemes are summarized in table 3.

The above results indicate that in comparison with other time advancement algorithms, SBDF2 provides less accuracy in solving plane channel flow problem in the same time step. The comparison between CNRK3 and SMRK2 method indicates that the proposed low-storage Runge-Kutta method do slightly reduces the total time. It looks that the use of Runge-Kutta algorithm for the solution of nonlinear term is time consuming. This modification decreases the overall time from 7.706e+04 to 6.593e+04 (just about 14.4%) which still is more than 2nd order Adams-Bashforth method in CNAB2.

The verification and accuracy of the presented analysis in rotational form with varied time step algorithm and different

Table 2. Specification of turbulent flow in different time advancement schemes in rotational

Scheme	Total Time (sec)	Wall-Shear Velocity(u_τ)	Standard Dev.	Dissipation (U+u)	Energy(u)	Energy(U+u)
CNAB2	2.381e+04	0.0436921	0.10455	5.13676	0.0193159	0.186586
CNRK3	6.593e+04	0.04384081	0.104836	5.06095	0.0195952	0.187419
SMRK2	7.706e+04	0.04368077	0.104915	5.03466	0.0194839	0.187097
SBDF2	3.065e+04	0.04304397	0.107202	5.908740	0.018696	0.1859460
SBDF3	2.137e+04	0.04397750	0.104729	4.98555	0.019676	0.187878
SBDF4	2.726e+04	0.04377228	0.105062	5.02415	0.0196434	0.187584

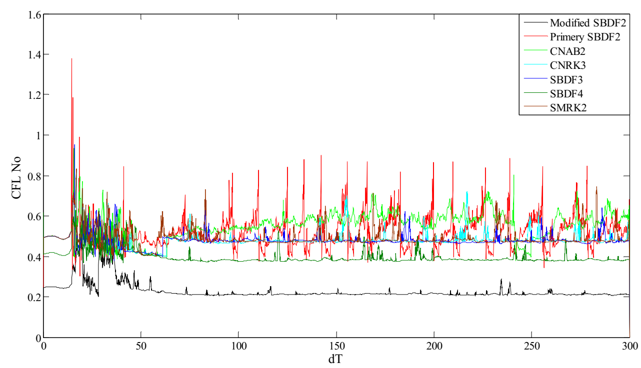

Fig. 5. variation of CFL number in different time advancement scheme

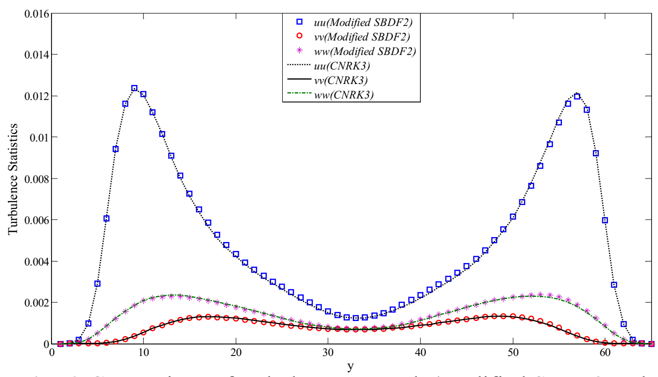

Fig. 6. Comparison of turbulence strength (modified SBDF2 and other time advancement scheme)

time advancement methods is validated in Fig. 7 with Kim et. al. results [7], in a fully developed turbulent channel flow and arbitrary time-integration schemes like SBDF3. Turbulence statistics profiles show excellent agreements between our calculation and Kim et. al. ones which are based on constant time stepping algorithm.

Also, it should be noticed that based on 128 × 65 × 128 grid points in this research, the uniform grid spacing in the stream-wise and span-wise directions are $\Delta x^{+} \approx 15$ and $\Delta z^{+} \approx 7.5$ respectively. In the vertical direction, a non-uniform mesh distribution is used, with the minimum grid spacing of $\Delta y^{+}_{min} \approx 0.2$ near the wall and the maximum of $\Delta y_{max} \approx 7.5$ spacing at the centerline of the channel, which are sufficient and acceptable due to Moser et. al. [24] and Moin & Mahesh [3]. In Kim et. al research the grid spacing are: $\Delta x^{+} \approx 12$, $\Delta y_{max} \approx 4.4$, $\Delta y^{+}_{min} \approx 0.05$ and $\Delta z^{+} \approx 7$ in the wall units within192 × 129 × 160 mesh points. It would obtain friction Reynolds number of $Re_{\tau} = 175$ that seems to be excessive for the under consideration Reynolds number.

In addition various discritization of nonlinear term with SBDF3 time advancement method by removing aliasing errors in X and Z direction are studied. The result of turbulence statistics in stream wise, normal and span wise directions are respectively shown in Figures 8 to 10. Table 4 reviews the results of overall CPU, wall shear velocity, normalized standard deviation of u in each method, and the normalized total energy of instantaneous velocities.

Table 4 shows minor difference between the computed values for energy, wall shear velocities and standard deviation of u component. Obviously, several algorithms like Skew-Symmetric method and combined Alternating method are more time consuming. It should be noticed that18 derivatives are required for the evaluation of skew-symmetric and convection form which need greatest computational cost. Divergence and alternating forms take 9 derivatives whereas only 6 derivatives are needed in the rotation form that seems more economical.

It is also important that the Linearized method is not a suitable suggestion due to instability problem in pseudo spectral method.

Table 3. The deviation of averaged values of turbulence statistics in different time advancement schemes

Turbulence Strength	SBDF2	Modified SBDF2	Averaged of other time stepping scheme
uu	1.94E-03 (156%)	2.63% (2.08E-04)	4.00 E-4 (3.36%) in SBDF4
vv	3.33E-03 (482%)	1.72% (2.03E-05)	6.02E-05 (8.69%) in CNAB2
ww	3.54E-03 (490%)	1.69% (3.96E-05)	1.21E-04 (5.28%) in SBDF4

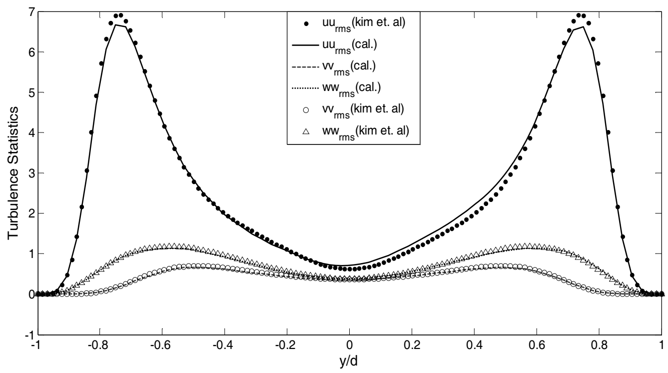

Fig. 7. Root-mean-square velocity fluctuations from SBDF3 and Kim et.al. results, symbols represent the data from Kim et.al. [4]

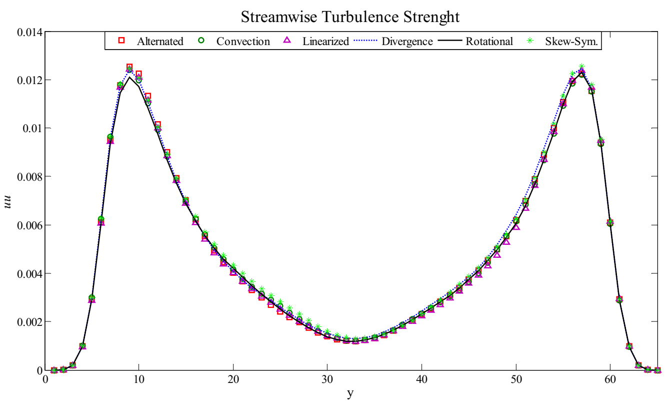

Fig. 8. Comparison of stream-wise turbulence strength at different version of nonlinear term in SBDF3 scheme

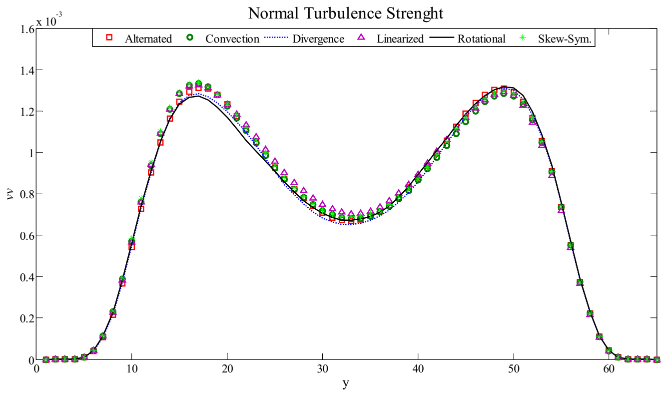

Fig. 9. Comparison of normal turbulence strength at different version of nonlinear term in SBDF3 scheme

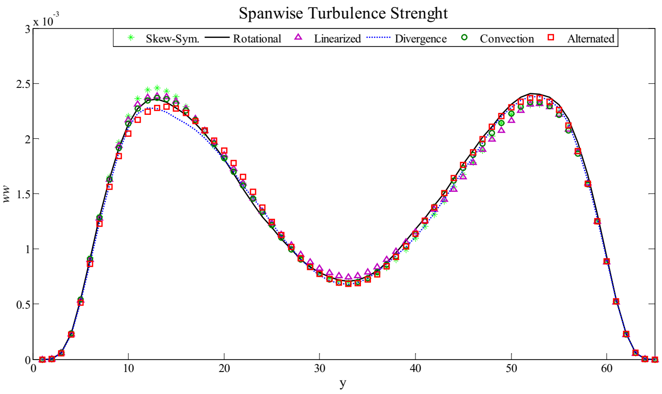

Fig. 10. Comparison of span-wise turbulence strength at different version of nonlinear term in SBDF3 scheme

Table 4. Specification of turbulent flow in different convection term scheme

Scheme	Time stepping scheme	Total Time (sec)	Wall-Shear Velocity(u_τ)	Standard Dev.	Energy(u)
Rotational	SBDF3	2.137e+04	0.04397750	0.104729	0.019676
Skew-Symmetric	SBDF3	5.65e+04	0.04417833	0.105915	0.0199884
Convection	SBDF3	3.278e+04	0.04407398	0.10466	0.019753
Divergence	SBDF3	3.126e+04	0.04397043	0.10503	0.0196342
Alternating	SBDF3	3.636e+04	0.04395179	0.104943	0.0196932
Linearized	SBDF3	1.874e+04	0.04408621	0.105426	0.0199947

6. Conclusion

The present work provides a complete and systematic study of various time advancement schemes and different expression of nonlinear terms for the three-dimensional incompressible Navier-Stokes equations to find the best framework for the analysis of a Poiseuille channel flow. A variable time stepping algorithm is proposed as a desirable modification to reduce the computation cost. Two common time-discretization strategies of splitting techniques and monolithic method are used. It is specifically shown that between the sampling techniques, the variable time stepping third order backward difference algorithm provides simultaneity more improvement in total CPU time and accuracy. Also the results demonstrate that low-storage Runge-Kutta method has less effect on total time reduction.

In nonlinear term explanation, against the result of other numerical methods such as [23] and [20], in a fully tau pseudo spectral method, the comparisons presented above demonstrated that the performance of rotation method is the same as skew symmetric and other form of discritization, whenever the aliasing errors are removed. It is concluded that based on economic consideration, the dealiased rotational form is the best choice and skew-symmetric scheme needs greatest computational cost without much accuracy improvement.

From practical point of view, the most desirable approach is a combination of third-order SBDF3 and rotational form in a variable time stepping algorithm. It is believed that this algorithm provides further cost and accuracy improvements in the process of simulation.

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