Technical Note:

Variation Theorems for Dynamic Analysis of 2D Structures

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Abstract: *In this paper, the Rayleigh's quotient and the inverse vector iteration method are presented. The latter approach helps to obtain the natural frequencies and mode shapes of a structure. Inverse vector iteration method with shifting enables to determine the higher modes. First some basic theorems of linear algebra are presented and extended to study the free vibration of structures. Then the variation theorems are presented for predicting the eigenvalues and eigenvectors of the modified structures. These theorems reduce the number of cycles of the iterations used for calculating the eigenvalues and eigenvectors of the modified structures. Finally, an example is solved to show the ability of the present approach.*

Keywords: *Structural variation theorems, Inverse vector iteration method, Rayleigh's quotient, Eigenvalues, Eigenvectors and Gram-Schmidt Orthogonalization*

1. Introduction

A major problem that often confronts a designer is the manner in which the natural frequencies and mode shapes of a structure change, when one or more of its members are varied. The repeated use of iteration methods to fulfill this objective can be cumbersome and time consuming.

The theorems of structural variation in static analysis predict the forces and displacements throughout a structure without the need of fresh analysis when the physical properties of one or more members are altered, or even its topology is changed due to removal of one or more of its elements, Majid and Elliot [1], Majid et al. [2], Saka [3-5], Kaveh and Najimi [6].

In dynamic analysis of structures, in order to predict the eigenvalues and eigenvectors of the modified structures, some researchers employed perturbation method, Lü et al. [7], Eldred et al. [8], and Liu and Chan [9]. Other researchers applied reanalysis techniques to obtain the eigenvalues and eigenvectors of the modified structures, Chen et al. [10,11], Chuarong and Yimin [12] and Kirsch [13].

One of the most efficient methods of dynamic analysis is inverse vector iteration method. In this method, selection of a suitable initial eigenvalue and eigenvector is very important in reducing the number of the cycles of iterations. The theorems of structural variation in dynamic analysis employ the eigenpairs properties and bounds of eigenvalues to decrease the number of cycles of the iterations when used for calculating the eigenvalues and eigenvectors of modified structures. The eigenpairs properties and the bounds of eigenvalues have been previously studied in [14-20].

In this paper, the inverse vector iteration method, together with eigenpair properties and bounds of eigenvalues are presented. Based of these properties, the procedure of variation theorem is proposed. Finally, one example is solved to show the capability of the present method compared to the iterative method and the reanalysis technique.

1.1 Inverse Vector Iteration Method

Finding the vibration properties - Natural frequencies and modes of a structure requires the solution of the matrix eigenvalue problem which is given by Eq. (1) or Eq. (2), Ref. [14]:

$$\mathbf{k}\phi = \lambda \mathbf{m}\phi \tag{1}$$

$$\det(\mathbf{k} - \lambda \mathbf{m}\mathbf{I}) = 0 \tag{2}$$

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where **m** and **k** are the mass matrix and stiffness matrix, respectively. λ_1 to λ_2 and ϕ_1 to ϕ_n are the eigenvalues and eigenvectors, respectively, where n is the order of **m** and **k**.

Inverse vector iteration method is often been used to solve these equations. In this method Rayleigh's quotient is employed [14]:

$$\lambda = \omega^2 = \frac{\phi^T k \phi}{\phi^T m \phi} \tag{3}$$

This procedure starts with the assumption of a starting iteration vector x_1 and consists of the following steps to be repeated for j=1,2,3,... until convergence occurs [14].

- Determine $\bar{x}_{j+1}$ by solving the algebraic equations

$$k\bar{x}_{j+1} = mx_j \tag{4}$$

- Obtain an estimate of the eigenvalue by evaluating Rayleigh's quotient

$$\lambda^{(j+1)} = \frac{\bar{x}_{j+1}^T k \bar{x}_{j+1}}{\bar{x}_{j+1}^T m \bar{x}_{j+1}} = \frac{\bar{x}_{j+1}^T m x_j}{\bar{x}_{j+1}^T m \bar{x}_{j+1}} \tag{5}$$

- Check the convergence by comparing two successive values of λ as

$$\frac{\left|\lambda^{(j+1)} - \lambda^{(j)}\right|}{\lambda^{(j+1)}} \le \text{tolerance} \tag{6}$$

- If the convergence criterion is not satisfied, normalize $\bar{x}_{j+1}$ as

$$x_{j+1} = \frac{\bar{x}_{j+1}}{(\bar{x}_{j+1}^T m \bar{x}_{j+1})^{0.5}} \tag{7}$$

and go back to the first step and carry out another iteration using the next j.

- Let *l* be the last iteration i.e., the iteration which satisfies Eq. (6). Then

$$\lambda_l = \lambda^{(l+1)} \quad \text{and} \quad \phi_l = \frac{\bar{x}_{l+1}}{(\bar{x}_{l+1}^T m \bar{x}_{l+1})^{0.5}} \tag{8}$$

Here x and $\bar{x}$ are the normalized and non-normalized eigenvectors of structure.

1.2 Evaluation of higher modes by using vector iteration with shifts

The inverse iteration procedure combined with the concept of "shifting" provides an effective means to improve the convergence rate of the iteration process, and to make it to converge to an eigenpair other than (λ_1,ϕ_1), Ref. [14].

Figure 1(a) shows the eigenvalue spectrum of Eq. (1). Introducing a shift μ in the origin of the eigenvalue axis (Fig. 1(b)) and defining $\bar{\lambda}$ as the shifted eigenvalue measured from the shifted origin, gives $\lambda = \bar{\lambda} + \mu$. Substituting this in Eq. (1) leads to

$$\bar{k}\phi = \bar{\lambda} m \phi \tag{9}$$

Where

$$\bar{k} = k - \mu m \quad \text{and} \quad \bar{\lambda} = \lambda - \mu \tag{10}$$

Here $\bar{k}$ is the shifted matrix used to evaluate the higher modes of the structure. The eigenvectors of the two eigenproblems (Eq. (1) and Eq. (9)) are identical. This is obvious because if a ϕ satisfies one equation, it will also satisfy the other. However, the eigenvalues $\bar{\lambda}$ of the shifted problem differ from the eigenvalues λ of the original problem by the shift μ (Eq. (9)). The spectrum of the shifted eigenvalues $\bar{\lambda}$ is also shown in Fig. 1(b) with the origin at μ. If μ was chosen between λ_n and λ_{n+1}, and μ is closer to λ_n than λ_{n+1}, the iteration of Eq. (9) will converge to $\bar{\lambda}_n$ and the iteration of Eq. (1) will converge to λ_n. On the other hand, if μ is closer to λ_{n+1} than λ_n, the iteration of Eq. (9) will converge to $\bar{\lambda}_{n+1}$ and the iteration of Eq. (1) will converge to λ_{n+1}. Thus the "shifting" concept enables one to compute any pair $(\lambda_n.\phi_n)$.

2. Variation Theorems

When the physical properties of one or more members are altered, the variation theorems are used to predict the eigenvalues and eigenvectors

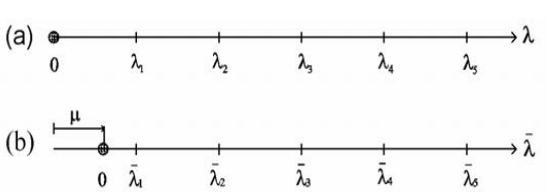

Fig. 1 (a) Eigenvalue spectrum; (b) Eigenvalue measured from a shifted origin [14].

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without the need of a fresh analysis.

2.1 Basic theorems and results

2.1.1 In equation det(**A**-λ**I**)=0 , if **A** be an n×n matrix with eigenvalues λ_1 to λ_n, then [15-18]:

$$\mathrm{tr}(A)=\sum_{i=1}^{n}\lambda_i \tag{11}$$

In equation (2) for the structural vibration, if **m** is diagonal and $m_{ii}\neq 0$, this relation is given by

$$\frac{k_{11}}{m_{11}}+\frac{k_{22}}{m_{22}}+....+\frac{k_{nn}}{m_{nn}}=\lambda_1+\lambda_2+...\lambda_n=\sum_{i=1}^{n}\lambda_i \tag{12}$$

Proof:

If Eq. (1) is pre-multiplied by $\mathbf{m}^{-1}$, then $\mathbf{m}^{-1}k\phi=\lambda\phi$.

Let $\mathbf{m}^{-1}k=\mathbf{B}$. Since **m** is a diagonal matrix, then $\mathbf{m}^{-1}$is also diagonal, therefore the diagonal of matrix is obtained by multiplying the diagonal entries of $\mathbf{m}^{-1}$ by those of k. i.e.

$$\text{trace }\mathbf{B}=m_{11}^{-1}k_{11}+m_{22}^{-1}k_{22}+....+m_{nn}^{-1}k_{nn}=\sum_{i=1}^{n}\lambda_i$$

Also

$$m_{11}^{-1}=\frac{\prod_{i=2}^{n}m_{ii}}{\prod_{i=1}^{n}m_{ii}}=\frac{1}{m_{11}} \qquad \text{and ,}$$

$$m_{22}^{-1}=\frac{\prod_{i=1,i\neq 2}^{n}m_{ii}}{\prod_{i=1}^{n}m_{ii}}=\frac{1}{m_{22}},\ .....,\ m_{nn}^{-1}=\frac{\prod_{i=1,i\neq n}^{n}m_{ii}}{\prod_{i=1}^{n}m_{ii}}=\frac{1}{m_{nn}}$$

$$\text{Then trace}=\frac{k_{11}}{m_{11}}+\frac{k_{22}}{m_{22}}+....+\frac{k_{nn}}{m_{nn}}=\sum_{i=1}^{n}\lambda_i\ .$$

2.1.2 In the equation det(**A**-λ**I**)=0, if **A** is an n×n matrix with eigenvalues λ_1 to λ_n , then [15-18]:

$$\det(A)=\lambda_1\lambda_2.....\lambda_n=\prod_{i=1}^{n}\lambda_i \tag{13}$$

In the structural vibration, this relation is given by

$$\lambda_1\lambda_2.....\lambda_n=\prod_{i=1}^{n}\lambda_i=\frac{\det(k)}{\det(m)} \tag{14}$$

Proof:

$$k\phi=\lambda m\phi \quad \Rightarrow \quad m^{-1}k\phi=\lambda\phi \quad \Rightarrow$$

$$\det(m^{-1}k)=\prod_{i=1}^{n}\lambda_i$$

and $\det(m^{-1}k)=\det(m^{-1})\times\det(k)$.

Also $\det(m^{-1})=\dfrac{1}{\det(m)}$.

Thus

$$\prod_{i=1}^{n}\lambda_i=\frac{\det(k)}{\det(m)}$$

When matrix **A** is reduced to **B** with eigenvalues $\lambda_1',\lambda_2',...,\lambda_n'$ then

$$\lambda_1'\lambda_2'.....\lambda_n'=\frac{\det(B)}{\det(A)}\times\lambda_1\lambda_2.....\lambda_n \tag{15}$$

In the structural vibration, if **k**, **m** with eigenvalues $\lambda_1,\lambda_2,...,\lambda_n$ are reduced to $\mathbf{k}'$, $\mathbf{m}'$ with eigenvalues $\lambda_1',\lambda_2',...,\lambda_n'$, respectively, then

$$\lambda_1'\lambda_2'.....\lambda_n'=\frac{\det(k')\det(m)}{\det(k)\det(m')}\times\lambda_1\lambda_2.....\lambda_n \tag{16}$$

From this concept, the first approximation of λ_1' to λ_n' can be formed by calculating as:

$$\alpha=\sqrt[n]{\frac{\det(k')\det(m)}{\det(k)\det(m')}} \tag{17}$$

$$\lambda_1'=\alpha\lambda_1 \text{ and } \lambda_2'=\alpha\lambda_2 \text{ and } ..... \lambda_n'=\alpha\lambda_n\ .$$

2.1.3 If λ_1 to λ_n ($\lambda_1\leq\lambda_2\leq...\leq\lambda_n$) are the eigenvalues and x is an assumed eigenvector, Rayleigh's quotient is between λ_1 to λ_n [14].

$$\lambda_1\leq\frac{x^Tkx}{x^Tmx}\leq\lambda_n \tag{18}$$

This property helps us to predict the first eigenvalue. Then in the new matrices $\mathbf{k}'$ and $\mathbf{m}'$ we have

$$\lambda_1'\leq\frac{x^Tk'x}{x^Tm'x} \tag{19}$$

resulting in the following properties:

A. If x is assumed as $x=[1\ 0\ 0\ 0\ 0]^t$, then

$$\lambda_1'\leq\frac{k_{11}'}{m_{11}'},\frac{k_{22}'}{m_{22}'},.....,\frac{k_{nn}'}{m_{nn}'} \tag{20}$$

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B. If x is assumed as $x=[1\ 1\ 1\ 1\ 1]^t$,

then $\lambda_1' \le \dfrac{\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}'}{\sum_{i=1}^{n} m_{ii}'}$.

2.1.4 Gram- Schmidt Orthogonalization Process

Let $\{\phi_1,...,\phi_n\}$ be a basis for a vector space ϕ. On order to obtain an orthonormal basis $\{u_1,...,u_n\}$, one should normalize each of the vectors $u_1,...,u_n$ as [15]:

$$u_1 = \phi_1$$
$$u_2 = \phi_2 - \text{proj}_{u_1}\phi_2$$
$$u_3 = \phi_3 - \text{proj}_{u_1}\phi_3 - \text{proj}_{u_2}\phi_3$$
$$\ldots\ldots\ldots\ldots$$
$$u_n = \phi_n - \text{proj}_{u_1}\phi_n - \text{proj}_{u_2}\phi_n - \ldots.. - \text{proj}_{u_{n-1}}\phi_n$$

$$\text{Where } \text{proj}_u\phi = \frac{\phi.u}{u.u}u \qquad (21)$$

2.1.5 Gerschgorin's theorem

In the equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$, if $A=[a_{ij}]_n$ is a matrix with eigenvalues λ_1 to λ_n, then [19-20]:

$$|\lambda - a_{ii}| \le \max\left\{\sum_{m=1,m\ne i}^{n}|a_{im}|\right\} \quad ,i=1, 2, \ldots, n \qquad (22)$$

In the free vibration of structures, this relation is given by

$$\left|\lambda - \frac{k_{ii}}{m_{ii}}\right| \le \max\left\{\sum_{m=1,m\ne i}^{n}\left|\frac{k_{im}}{m_{ii}}\right|\right\} \quad , i= 1, 2,\ldots,n \qquad (23)$$

$\frac{k_{ii}}{m_{ii}} \ge 0$, then this equation can be written as

$$\min\left\{\frac{k_{ii}}{m_{ii}} - \sum_{m=1,m\ne i}^{n}\frac{k_{im}}{m_{ii}}\right\} \le \lambda \le \max\left\{\sum_{m=1}^{n}\left|\frac{k_{im}}{m_{ii}}\right|\right\} \quad ,i= 1,2,...,n \qquad (24)$$

One of the results of Gerschgorin's theorem is [20]:

$$|\lambda| \le n \times \max\{a_{ij}\} \qquad (25)$$

This relation is extended to the free structural vibration as

$$|\lambda| \le n \times \max\left\{\frac{k_{ij}}{m_{ii}}\right\} \qquad (26)$$

2.2 Method for the solution of modified structure

The initial selection of eigenvalue and eigenvector is one of the problems of the inverse vector iteration method. In the iteration method, the value of the shift is also very important. For example, for evaluating the second mode, if the shift is taken near the first mode than the second mode, it converges to the first mode, and if it is not near enough to the second mode, the convergence rate will be reduced.

In variation method, first by using the eigenvalue and eigenvector of the initial structure, and mass and stiffness matrices of the modified structure, the eigenvalue and eigenvector are guessed. Then using these eigenvalue and eigenvector in the iteration process, the eigenvalue and eigenvector of the modified structure are calculated.

For the first mode, first by substituting the eigenvector of the initial structure, and the mass and stiffness matrices of the modified structure in Rayleigh's quotient, the eigenvalue of the modified structure should be guessed. Then by using this eigenvalue the corresponding eigenvector is obtained. Substituting these assumed eigenvalue and eigenvector in the iteration method the eigenvalue and eigenvector of the modified structure are obtained.

For higher modes, first by using the eigenvector of the initial structure, the eigenvector of the modified structure is guessed. Then by using this eigenvector, the quantity of shift and eigenvalue is obtained. Substituting the assumed eigenvector, shift, and eigenvalue in the iteration method eigenvalue and eigenvector of the modified structure are obtained.

We can see the rate of convergence is improved. In examples, the cycles of iteration are not needed and usually it converges in the first cycle.

2.2.1 Determination of first mode eigenvalue and eigenvector

If the mass and stiffness matrices shown in Fig. (2), are varied from $\mathbf{k}$ and $\mathbf{m}$ to $\mathbf{k}'$ and $\mathbf{m}'$, respectively, the eigenvalue and eigenvector of the first mode is reduced from λ_1 and ϕ_1 to λ_1' and ϕ_1', respectively. By using the above theorems, the eigenvalue and eigenvector of the first mode are determined. The algorithm is as follows:

- Substitute the eigenvector of the first mode of

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initial structure to the Rayleigh's quotient and determine λ'_{11}.

$$\lambda'_{11} = \frac{x^T k' x}{x^T m' x} \tag{27}$$

- Determine the determinant of the mass and stiffness matrices of initial and modified structures. Then by using Eq. (17) calculate α

- Determine λ'_{12} as

$$\lambda'_{12} = \alpha\lambda_1 \tag{28}$$

- If $\lambda'_{11} \leqq \lambda'_{12}$, then $\lambda'_1 = \lambda'_{11}$. If $\lambda'_{12} \leqq \lambda'_{11}$, and if the eigenvalue is less than the initial state, then consider $\lambda'_1 = \lambda'_{11}$. In the unusual variation, if $\lambda'_{12} \leqq \lambda'_{11}$ and the eigenvalue is increased compared to the initial state, then put $\lambda'_1 = \lambda'_{12}$.

- Check λ'_1 with k'_{11}/m'_{11}, k'_{22}/m'_{22},..., k'_{nn}/m'_{nn}. λ'_1 should be less than these amounts (this control is not important in usual variations).

- By using λ'_1 and $\mathbf{k}'$ and $\mathbf{m}'$, determine the corresponding eigenvector.

- Put the eigenvalue and eigenvector in the iteration cycle.

Note 1: In the unusual variations, when λ'_{11} is too high, one can calculate the eigenvector for λ'_{12} and put it in Rayleigh's quotient and detemine λ'_1.

Note 2: In the lumped mass structures, shown in Fig. (2), when the eigenvalue is determined, the corresponding eigenvector is obtained as

$$x'_1 = \left\{ \begin{array}{c} \phi_1 = 1 \\ \phi_2 = -\dfrac{(k'_{11} - m'_{11}\lambda'_1)\phi_1}{k'_{12}} \\ \phi_3 = -\dfrac{k'_{21}\phi_1 + (k'_{22} - m'_{22}\lambda'_1)\phi_2}{k'_{23}} \\ . \\ \phi_i = -\dfrac{k'_{(i-1)(i-2)}\phi_{(i-2)} + (k'_{(i-1)(i-1)} - m'_{(i-1)(i-1)}\lambda'_1)\phi_{(i-1)}}{k'_{(i-1)i}} \end{array} \right\} \tag{29}$$

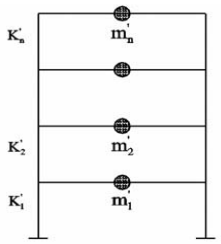

Fig. 2 The lumped mass model of a structure

2.2.2 Evaluation of higher modes

If the mass and stiffness matrices, shown in Fig. (2), are varied from **k** and **m** to $\mathbf{k}'$ and $\mathbf{m}'$, respectively, then the eigenvalues will be reduced from $\lambda_2, \lambda_3, ..., \lambda_n$ to $\lambda'_2, \lambda'_3, ..., \lambda'_n$ and the eigenvectors will be reduced from $\phi_2, \phi_3, ..., \phi_n$ to $\phi'_2, \phi'_3, ..., \phi'_n$, respectively.

Now, by employing $\{u_1, ..., u_n\}$ the eigenvalues and eigenvectors of higher modes are obtained. These modes are evaluated using the Gram-Schmidt orthogonalization and the eigenvectors of the initial structure are as given in equation (21).

2.2.2.1 Determination of second mode eigenvalue and eigenvector

Using u_2 and the first eigenvector of the modified structure, ϕ'_2 is approximated as

$$\phi'_2 = u_2 - \text{proj}_{\phi'_1} u_2 \tag{30}$$

Then substituting ϕ'_2 in $\frac{\phi'^T_2 K' \phi'_2}{\phi'^T_2 m' \phi'_2}$ and guessing the quantity of shift and λ'_2, the iteration is performed , and the eigenpairs for the second mode is calculated.

2.2.2.2 Determination of the eigenvalue and eigenvector for the n^{th} mode

Using u_n and the first to $(n-1)^{th}$ eigenvectors of the modified structure, ϕ'_n is evaluated as

$$\phi'_n = u_n - \text{proj}_{\phi'_1} u_n - \text{proj}_{\phi'_2} \phi_n - ..... - \text{proj}_{\phi'_{n-1}} \phi_n \tag{31}$$

Substituting ϕ'_n in $\frac{\phi'^T_n K' \phi'_n}{\phi'^T_n m' \phi'_n}$ and guessing the quantity of the shift and λ'_n, the iteration cycle is performed and the eigenpairs for the n^{th} mode is obtained.

3. Numerical Examples

Consider the structure of Fig. (3), where m=0.259kip-sec^2/in and k=168kips/in. This structure has 5 degrees of freedom. The mass and stiffness matrices of this structure are given by

$$m = \begin{bmatrix} 0.259 & 0 & 0 & 0 & 0 \\ 0 & 0.259 & 0 & 0 & 0 \\ 0 & 0 & 0.1295 & 0 & 0 \\ 0 & 0 & 0 & 0.1295 & 0 \\ 0 & 0 & 0 & 0 & 0.0863 \end{bmatrix} \quad \det(m) = 9.708449 \times 10^{-5}$$

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$$K=\begin{bmatrix} 336 & -168 & 0 & 0 & 0 \\ -168 & 298.67 & -130.67 & 0 & 0 \\ 0 & -130.67 & 224 & -93.33 & 0 \\ 0 & 0 & -93.33 & 149.33 & -56 \\ 0 & 0 & 0 & -56 & 56 \end{bmatrix} \quad \det(k)=1.928\times10^{10}$$

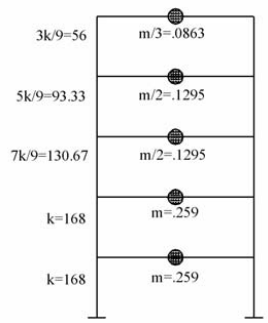

Fig. 3 The primary structure

Assuming $\lambda=1$ and $x=[1\ 1\ 1\ 1\ 1]^t$, the first mode is determined in 4 cycles of iteration as $\lambda_1=90.31$ and $\omega_1=9.5032$. The eigenvector is calculated as

$$x_1=[0.4568\ \ 0.8427\ \ 1.1941\ \ 1.5331\ \ 1.7787]^t$$

Considering a *"shift =300"* and $x=[1\ 1\ 1\ 1\ 1]^t$, the second mode is determined in 15 cycles of iteration. When the shift is taken as 400, then the second mode is determined in 4 cycles of iteration as $\lambda_2=433.455$ and $\omega_2=20.8196$. The corresponding eigenvector is calculated as

$$x_1=[0.879\ \ 1.171\ \ 0.527\ \ -0.6695\ \ -2.016]^t$$

For a shift equal to 250, it converged to the first mode, since 250 is nearer to 90.31 than 433.455. Thus, the importance of the quantity of shift becomes apparent.

In the second state, the mass and stiffness matrices of this structure is given by

$$m'=\begin{bmatrix} 0.1295 & 0 & 0 & 0 & 0 \\ 0 & 0.1295 & 0 & 0 & 0 \\ 0 & 0 & 0.1295 & 0 & 0 \\ 0 & 0 & 0 & 0.1295 & 0 \\ 0 & 0 & 0 & 0 & 0.0863 \end{bmatrix}$$

$$K'=\begin{bmatrix} 280 & -130.67 & 0 & 0 & 0 \\ -130.67 & 224 & -93.33 & 0 & 0 \\ 0 & -93.33 & 186.67 & -93.33 & 0 \\ 0 & 0 & -93.33 & 149.33 & -56 \\ 0 & 0 & 0 & -56 & 56 \end{bmatrix}$$

The first and second eigenvalues and eigenvectors of the modified structure are presented in Table 1 and Table 2, respectively.

If shift is selected as 500, then only 6 cycles of iteration was needed. If the shift is selected as 300, the iteration converged to the first eigenvalue after 10 cycles, since 300 is closer to 84.148 than 577.472. Thus the importance of the quantity of shift becomes more apparent.

4. A Comparative Study

In this section, a comparative study of the variation theorem, iterative method and reanalysis method is made. The variation theorems and iterative method are based on inverse vector iteration method. In the variation method, we are concerned with the role of the first assumption of the eigenvalue and eigenvector to decrease the cycles of iterations, while in the reanalysis method [13], the one tries to decrease the order (size) of matrices. When the order of matrices are decreased, the rate of the convergence will be modified and the accuracy will also be reduced.

A comparative study of these methods is presented in Table 3. The computational time of these methods for the first and second mode shapes are also compared in Fig. (4) and Fig. (5), respectively. It is obvious that the performance of the present method is better than both the iterative approach and the reanalysis technique. The reanalysis technique converges faster than the iterative method, however, the accuracy of the iterative method is higher than the reanalysis technique.

5. Concluding Remarks

- In the inverse vector iteration method for the first mode, the quality of the initial eigenvalue and eigenvector is important in order to reduce the number of cycles of the iteration. The variation method applies to the eigenpairs properties to provide a suitable approximation to these entities. Therefore, this method reduces the number of cycles of the iterations, significantly.

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Table 1 First mode of the modified structure

Iterative method

Iteration (j)	x_j	λ	x_{j+1}
1	$[1 \quad 1 \quad 1 \quad 1 \quad 1]^t$	85.8530	$[0.481 \quad 0.913 \quad 1.352 \quad 1.627 \quad 1.810]^t$
.	.	.	.
.	.	.	.
.	.	.	.
4	$[0.400 \quad 0.822 \quad 1.316 \quad 1.657 \quad 1.902]^t$	84.1478	$[0.399 \quad 0.821 \quad 1.316 \quad 1.657 \quad 1.903]^t$
5	$[0.399 \quad 0.821 \quad 1.316 \quad 1.657 \quad 1.903]^t$	84.1478	$[0.398 \quad 0.820 \quad 1.315 \quad 1.657 \quad 1.904]^t$

Present method

Substituting x_1 in the Rayleigh's quotient:

$$\lambda'_{11} = \frac{x^T \mathbf{k}' x}{x^T \mathbf{m}' x} = 86.56$$

Eigenvalue is reduced compared to that of the initial structure, $\lambda'_1 = \lambda'_{11} = 86.56$

$$x'_1 = [1 \quad 2.057 \quad 3.290 \quad 4.128 \quad 4.698]^t$$

Iteration (j)	x_j	λ	x_{j+1}
1	$[1 \quad 2.057 \quad 3.290 \quad 4.128 \quad 4.698]^t$	84.148	$[0.399 \quad 0.819 \quad 1.317 \quad 1.658 \quad 1.902]^t$

Reanalysis method [13]

S is considered as 2

$$C = k^{-1}(\Delta k - m')$$

$$C = \begin{bmatrix} -0.1119 & -0.0008 & -0.0007 & -0.0008 & -0.0005 \\ 0.1103 & -0.2237 & -0.0014 & -0.0015 & -0.0010 \\ 0.1103 & 0.0620 & -0.2881 & -0.0025 & -0.0017 \\ 0.1103 & 0.0620 & -0.2881 & -0.0039 & -0.0026 \\ 0.1103 & 0.0620 & -0.2881 & -0.0039 & -0.0042 \end{bmatrix}$$

$$r_1 = \phi_1 = [0.4568 \quad 0.8427 \quad 1.1941 \quad 1.5331 \quad 1.7787]^t$$

$$r_2 = -Cr_1 = [0.2147 \quad 0.5654 \quad 0.9744 \quad 0.9892 \quad 1]^t$$

$$r_B = \begin{bmatrix} 0.4568 & 0.8427 & 1.1941 & 1.5331 & 1.7787 \\ 0.2147 & 0.5654 & 0.9744 & 0.9892 & 1 \end{bmatrix}^t$$

$$K_R = r_B^t k' r_B = \begin{bmatrix} 76.261 & 46.374 \\ 46.374 & 38.605 \end{bmatrix}$$

$$M_R = r_B^t m' r_B = \begin{bmatrix} 0.881 & 0.575 \\ 0.575 & 0.383 \end{bmatrix}$$

$$K_R y = \lambda M_R y$$

Then the first eigenvalue is obtained as $\lambda = 85.231$

- For higher modes, the initial values of the eigenvalue and eigenvector, and shift are important to decrease the number of cycles. If the shift is not near enough to the desired eigenvalue, the iteration may converge to another eigenvalue. The variation method provides a good approximation to these entities. For evaluating higher modes, this method reduces the number of cycles of iteration considerably. The iteration usually converges in the first cycle of iteration.

- The present method is faster than the iterative approach. The reanalysis techniques provide approximate results while the present variation method, results in more accurate results and can

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Table 2 Second mode of the modified structure

Iterative method (shift=400

Iteration (j)	x_j	λ	x_{j+1}
1	$[1 \quad 1 \quad 1 \quad 1 \quad 1]^t$	237.770	$[0.334 \quad 0.213 \quad -0.596 \quad -1.593 \quad -2.648]^t$
2	$[0.334 \quad 0.213 \quad -0.596 \quad -1.593 \quad -2.648]^t$	364.624	$[1.039 \quad 1.733 \quad 1.669 \quad 0.884 \quad -0.330]^t$
.	.	.	.
.	.	.	.
.	.	.	.
13	$[0.990 \quad 1.556 \quad 1.102 \quad -0.232 \quad -2.140]^t$	577.472	$[0.989 \quad 1.552 \quad 1.097 \quad -0.239 \quad -2.149]^t$
14	$[0.989 \quad 1.552 \quad 1.097 \quad -0.239 \quad -2.149]^t$	577.472	$[0.989 \quad 1.554 \quad 1.100 \quad -0.235 \quad -2.144]^t$

Present method

$$u_1 = \phi_1$$

$$u_2 = \phi_2 - \text{proj}_{u_1}\phi_2 = [1.0298 \quad 1.4492 \quad 0.9212 \quad -0.1633 \quad -1.4288]^t$$

$$\phi_2' = u_2 - \text{proj}_{\phi_1} u_2 = [1.038 \quad 1.4654 \quad 0.9473 \quad -0.1306 \quad -1.3912]^t$$

$$\text{shift} = \frac{\phi_2'^{T}\mathbf{k}'\phi_2'}{\phi_2'^{T}\mathbf{m}'\phi_2'} = 579.2$$

Iteration (j)	x_j	λ	x_{j+1}
1	$[1.238 \quad 1.748 \quad 1.131 \quad -0.155 \quad -1.658]^t$	577.472	$[0.989 \quad 1.553 \quad 1.100 \quad -0.235 \quad -2.145]^t$

Reanalysis method [13]

S is considered as 2

$$C = k^{-1}(\Delta k - m')$$

$$C = \begin{bmatrix} -0.1119 & -0.0008 & -0.0007 & -0.0008 & -0.0005 \\ 0.1103 & -0.2237 & -0.0014 & -0.0015 & -0.0010 \\ 0.1103 & 0.0620 & -0.2881 & -0.0025 & -0.0017 \\ 0.1103 & 0.0620 & -0.2881 & -0.0039 & -0.0026 \\ 0.1103 & 0.0620 & -0.2881 & -0.0039 & -0.0042 \end{bmatrix}$$

$$\bar{r}_0 = \phi_2 = [0.8790 \quad 1.1710 \quad 0.5270 \quad -0.6695 \quad -2.0160]^t$$

$$\bar{r}_1 = [0.3983 \quad 0.8203 \quad 1.3153 \quad 1.6569 \quad 1.9037]^t$$

$$\alpha_1 = \bar{r}_1^{t} m' r_0 = -0.2154$$

$$r_0 = \bar{r}_0 - \alpha_1 \bar{r}_1$$

$$r_0 = [0.9648 \quad 1.3477 \quad 0.8103 \quad -0.3127 \quad -1.6060]^t$$

$$r_1 = r_0 = [0.9648 \quad 1.3477 \quad 0.8103 \quad -0.3127 \quad -1.6060]^t$$

$$r_2 = -Cr_1 = [0.5589 \quad 1 \quad 0.2058 \quad 0.1959 \quad 0.1831]^t$$

$$r_B = \begin{bmatrix} 0.9648 & 1.3477 & 0.8103 & -0.3127 & -1.6060 \\ 0.5589 & 1 & 0.2058 & 0.1959 & 0.1831 \end{bmatrix}^t$$

$$K_R = r_B^t k' r_B = \begin{bmatrix} 396.488 & 144.392 \\ 144.392 & 130.965 \end{bmatrix}$$

$$M_R = r_B^t m' r_B = \begin{bmatrix} 0.676 & 0.233 \\ 0.233 & 0.183 \end{bmatrix}$$

$$K_R y = \lambda M_R y$$

Then the second eigenvalue is obtained as $\lambda = 581.46$

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Table 3 Comparison of variation theorems, iterative method and reanalysis approach

	Iterative method	Present method	Reanalysis method [13]
Base of method	This method is based on the inverse vector iteration.	This method is based on inverse vector iteration.	Reduction in the order of matrices is the basis of this method.
Abilities	This method is an accurate method.	This method is an accurate approach. This method is also faster than iterative method.	This method is an approximate approach.
Weak points (disabilities)	In evaluation of higher modes, if shift is not selected adequately, this method converges after more cycles, and the procedure becomes time consuming	---	The accuracy of this method is depends on the order of reduction.

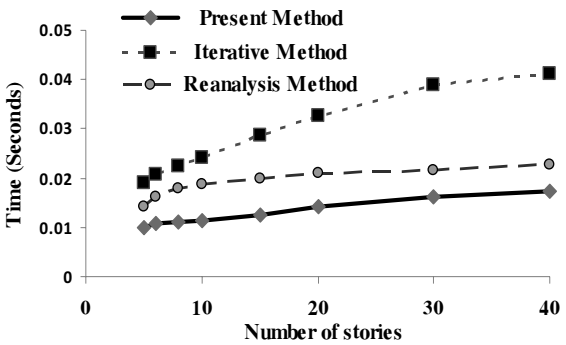

Fig. 4 Comparison of computational time for the first mode shape

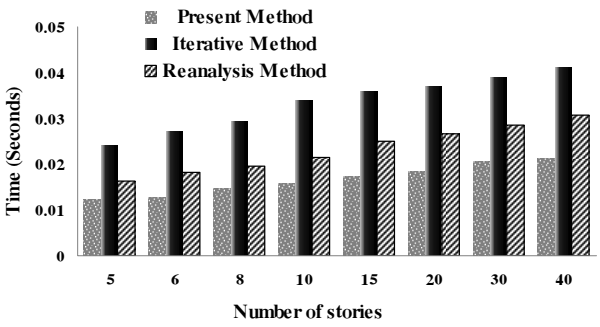

Fig. 5 Comparison of computational time for the second mode shape

be categorized as an exact approach.

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