International Journal of Civil Engineering

Structural Engineering

# New graph products for configuration processing

A. Kaveh[1,*], S. Beheshti[2]

Received: March 2012, Revised: October 2012, Accepted: November 2012

**Abstract**

*For the analysis of structures, the first step consists of configuration processing followed by data generation. This step is the most time consuming part of the analysis for large-scale structures. In this paper new graph products called triangular and circular graph products are developed for the formation of the space structures. The graph products are extensively used in graph theory and combinatorial optimization, however, the triangular and circular products defined in this paper are more suitable for the formation of practical space structural models which can not be generated easily by the previous products. The new products are employed for the configuration processing of space structures that are of triangular or a combination of triangular and rectangular shapes, and also in circular shapes as domes and some other space structural models. Cut out products are other new types of graph products which are defined to eliminate all of the connected elements to the considered node to configure the model or grid with some vacant panels inside of the model. The application of the presented graph products can be extended to the formation of finite element models.*

**Keywords:** *Triangular and circular graph products, Configuration processing, Generators; Space structures.*

## 1. Introduction

For a large system, configuration processing is one of the most time-consuming parts of the analysis. Different methods have been developed for configuration processing and data generation, among which one may refer to the formex algebra of Nooshin [1,2] (see also Nooshin et al. [3] Nooshin & Disney [4]), set theoretical approach of Behravesh et al. [5], and graph theoretical methods of Kaveh [6,7]. In all these methods a submodel is expressed in algebraic form and then functions are utilized for the formation of the entire model. The main functions employed consist of translation, rotation, reflection and projection, or their combinations. Four undirected graph products and four directed graph products are employed for the formation of structural models by Kaveh and Koohestani [8]. Weighted product graphs are also developed for this purpose by Kaveh and Nouri [9].

There are many other references in the field of data generation; however, most of them are prepared for specific classes of problems. For example, many algorithms have been developed and successfully implemented for mesh generation, a complete review of which may be found in a paper by Thacker [10], and in books by Thomson et al. [11] and Liseikin [12].

On the other hand, many structural models can be viewed as the graph products of two or three subgraphs, known as their generators. Many properties of structural models can be obtained by considering the properties of their generators. This simplifies many complicated calculations, particularly in relation with eigensolution of regular structures, as shown by Kaveh and Rahami [13,14].

In this paper, weights are assigned to the nodes and members of the generators to form configurations which cannot be generated using the four previously existing graph products [15]. These weighted products are especially suitable for the formation of the models of space structures. The present method can be used in the formation of other systems, and the application can easily be extended to the formation of finite element models.

## 2. Definitions from graph theory

A graph S(V,E) consists of a set of elements, V(S), called nodes and a set of elements, E(S), called members, together with a relation of incidence which associates two distinct nodes with each member, known as its ends. When weights are assigned to the members and nodes of a graph, then it becomes a weighted graph. Two nodes of a graph are called adjacent if these nodes are the end nodes of a member. A member is called

* *Corresponding Author: alikaveh@iust.ac.ir*
*1 Professor, Centre of Excellence for Fundamental Studies in Structural Engineering, Iran University of Science and Technology, Tehran, Iran*
*2 Graduate Student, School of Civil Engineering, Iran University of Science and Technology, Tehran, Iran*

incident with a node if that node is an end node of the member. The degree of a node is the number of members incident with the node. A subgraph $S_i$ of a graph $S$ is a graph for which $V(S_i) \subseteq V(S)$ and $E(S_i) \subseteq E(S)$, and each member of $S_i$ has the same ends as in S. A path graph $P$ is a simple connected graph with $V(P) = E(P)+1$ that can be drawn in a way that all of its nodes and members lie on a single straight line. A path graph with $n$ nodes is denoted as $P_n$, and a weighted path is shown by $PW_n$. A cycle graph $C$ is a simple connected graph with $V(C) = E(C)$ that can be drawn such that all of its nodes and members lie on a cycle. A cycle graph with n nodes is shown by $C_n$, and a weighted cycle is denoted by $CW_n$. For further definitions the reader may refer to Kaveh [7, 16, 17, 18, 19].

## 3. Extension of classic graph products

In this section, simple graph products are extended to new triangular and circular graph products for efficient use in structural mechanics.

### *3.1. Cartesian product of graphs:*

Some of the structures with regular configurations can be expressed as the Cartesian product of two or more subgraphs. After the formation of the nodes of such a graph according to the nodes of the generators, a member should be added between two typical nodes $(U_i, V_j)$ and $(U_k, V_l)$, as shown in Fig. 1, if any of the following conditions are satisfied:

$$\textit{If } \{[U_i = U_k \ \textit{ and } \ V_jV_l \in E(K)] \\ \textit{or } [V_j = V_l \ \textit{ and } \ U_iU_k \in E(H)]\}$$

It should be noted that $E(K)$ is the set of elements of subgraph $K$ and $E(H)$ is the elements of the subgraph $H$, and $U_i$ , $U_k$ are the nodes of subgraph $H$ and $V_j$ , $V_l$ are the nodes of subgraph $K$.

Two examples of this graph product are shown in Fig. 1.

### *3.2. Strong Cartesian product of graphs*

Some other structures with regular configurations can be expressed as the strong Cartesian product of two or more graphs. After the formation of the nodes of such a graph a member should be added between two typical nodes $(U_k, V_l)$ and $(U_i, V_j)$ if any of the following conditions are satisfied:

$$\textit{If } \{[U_i = U_k \ \textit{ and } \ V_jV_l \in E(K)] \\ \textit{or } [U_iU_k \in E(H) \ \textit{ and } \ V_j = V_l] \\ \textit{or } [U_iU_k \in E(K) \textit{ and } \ V_jV_l \in E(H)]\}$$

Two examples of strong Cartesian product are shown in Fig. 2. The configuration in Fig. 2(a) is the product of two paths and the one in Fig. 2(b) is the product of a cycle and a path and path and path.

### *3.3. Direct product of graphs*

Some other structures with regular configurations can be expressed as the direct product of two or more graphs. After the formation of the nodes of such a graph a member should be added between two typical nodes $(U_k, V_l)$ and $(U_i, V_j)$ if the following conditions are satisfied:

$$\textit{If } \{[U_iU_k \in E(K) \textit{ and } V_jV_l \in E(H)]\}$$

The direct product of two paths $P_3$ and $P_4$ and the product graph of $C_4$ and $P_4$ are shown in Fig. 3.

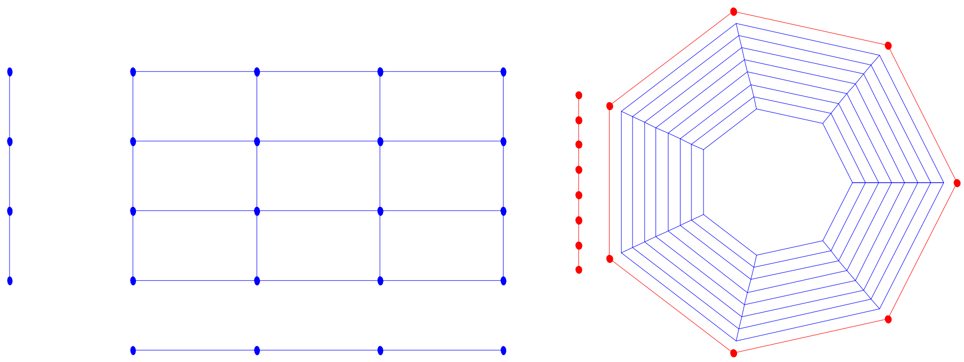

**Fig. 1** Cartesian graph products

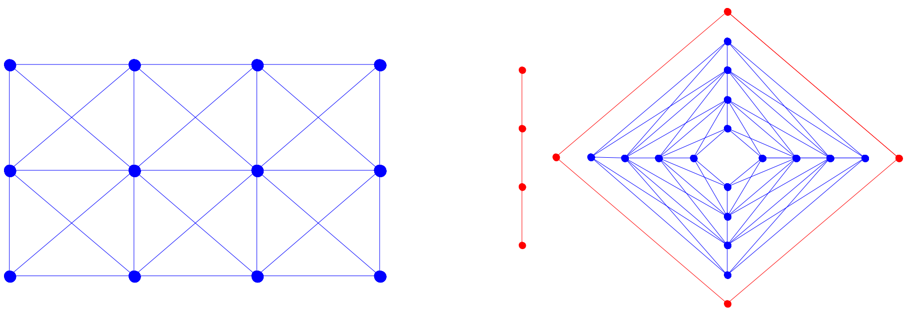

**Fig. 2** Strong cartesian graph products

## 4. Definition of triangular graph products

Since the classic graph products are only used as the adjacency conditions in the generators, so these products are plain and can form only simple shapes and models. For configuring other groups of the shapes and models, it is necessary to assign other conditions and definitions to the previous mentioned generators and also to the conditions of the products. Assigning weights to the nodes and elements can be considered as new additions to the conditions and definitions. For procreating the new graph products and new shapes, Kaveh and Nouri [9] generated new models in the forms of the classic products with assigning weights to the nodes and elements of the generators.

With a more profound insight it can be conceived that the concept of assigning weights to the nodes and defining new conditions according to these weights, can have a wide range and can make it possible to generate many other different shapes which are impossible to be generated by using classic products. In other words, the new definitions obtained by defining nodal weights, new variables become available to consider the adjacency conditions of the nodes in the product graph.

Using the existing graph products, only models in the form of rectangular or circular shapes can be generated with no center nodes. For generating other types of the shapes, either one should use a trans-figure of an existing product graph or should define new graph products. Now we define a new graph product for generating the triangular and a wide range of polygon shapes. By using this product it is possible to generate triangular, trapezoid, parallelogram and also many combinations of triangular and rectangular shapes.

### 4.1. New method of assigninging weigths to nodes and elements

In triangular products according to the new implemented definitions, the weights of the nodes of the product graph are equal to the summation of the weights of the nodes of the corresponding generators, Fig. 4. It is clear that assigning the weights to the nodes of a product graph according to the weights of the generators can have different types such as sum, difference, product, etc. After assigning the weights to the nodes of the product graph, some conditions should be imposed to assess the adjacency of the nodes. First $W_{MAX}$ is defined as the maximum weight of the subgraphs. For example, for two generators with the weights [1 2 3 4 5] and [2 3 2 3 4], $W_{MAX}$ is equal to 5. Considering the significance of $W_{MAX}$ and using $W_{i,j}(i,j)$ and $W_{i,j}(k,l)$, the new conditions are implemented.

### 4.2. Triangular Cartesian graph product

After the formation of the weights of the nodes in the product graph, a member should be added between two typical nodes $(U_k, V_l)$ and $(U_i, V_j)$ if the following conditions are satisfied:

Weights of the product graph nodes:

$W_{k,l} = W_i\,(k) + W_j\,(l)$ , $W_{i,j} = W_i\,(i) + W_j\,(j)$

Weights of the nodes of generators: $W_i$ , $W_j$

*If* $\{[[U_i = U_k \text{ and } V_j V_l \in E(K)] \text{ or } [U_i U_k \in E(H) \text{ and } V_j = V_l]]$
$\text{and } [Wi,j\ ,\ Wk,l \ \leq W_{\max}]]$
$\text{or } [[U_i U_k \in E(K) \text{ and } V_j V_l \in E(H)] \text{ and } [W_{i,j}\ ,\ W_{k,l} = W_{\max}]]\}$

Three examples of grids generated by these relationships are shown in Fig. 5. For each configuration, the corresponding weighted generators are also depicted.

### 4.3. Triangular strong Cartesian graph products

In this product both diagonal elements in all of the panels are also present. After the formation of weights of the nodes in the product graph, a member should be added between two typical nodes $(U_k, V_l)$ and $(U_i, V_j)$ if the following conditions are satisfied:

Weights of the nodes: $W_{k,l} = W_i\,(k) + W_j\,(l)$ , $W_{i,j} = W_i\,(i) + W_j\,(j)$

*If* $\{[[U_i = U_k \text{ and } V_j V_l \in E(K)] \text{ or } [U_i U_k \in E(H) \text{ and } V_j = V_l]]$
$\text{or } [[U_i U_k \in E(K) \text{ and } V_j V_l \in E(H)] \text{ and } [W_{i,j}\ ,\ W_{k,l} = W_{\max}]]\}$

Three examples of grids generated by these relationships are illustrated in Fig. 6. For each configuration, the corresponding weighted generators are also shown.

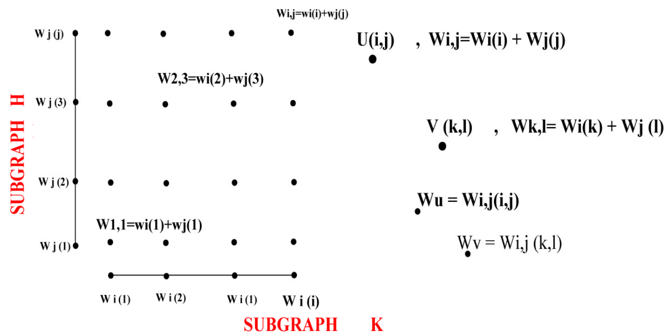


**Fig. 4** Assigning weights to the nodes of a product graph

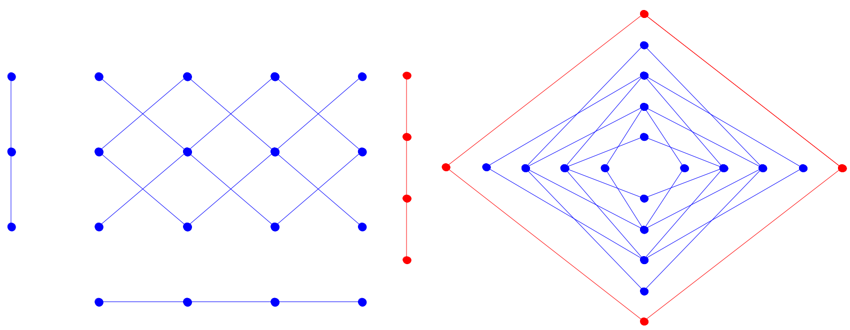

**Fig. 3** Direct graph products

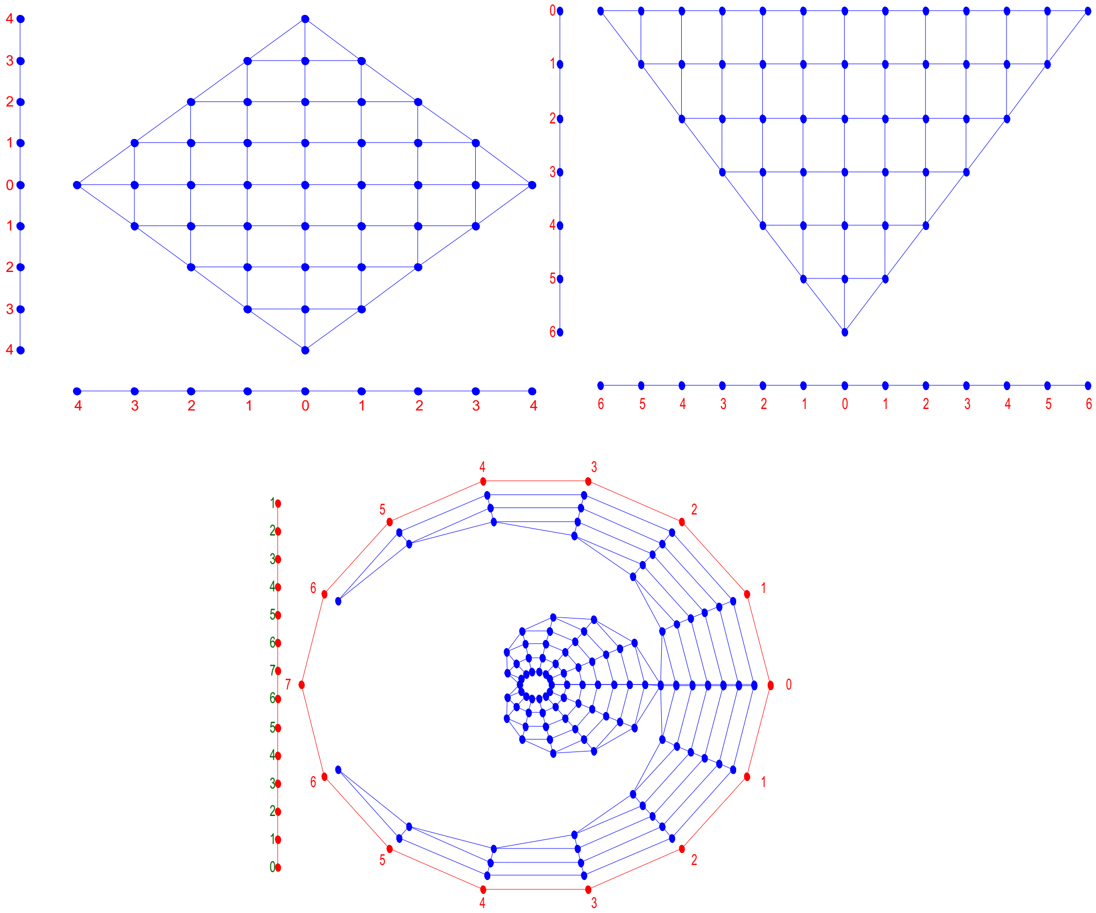

**Fig. 5** Triangular graph products

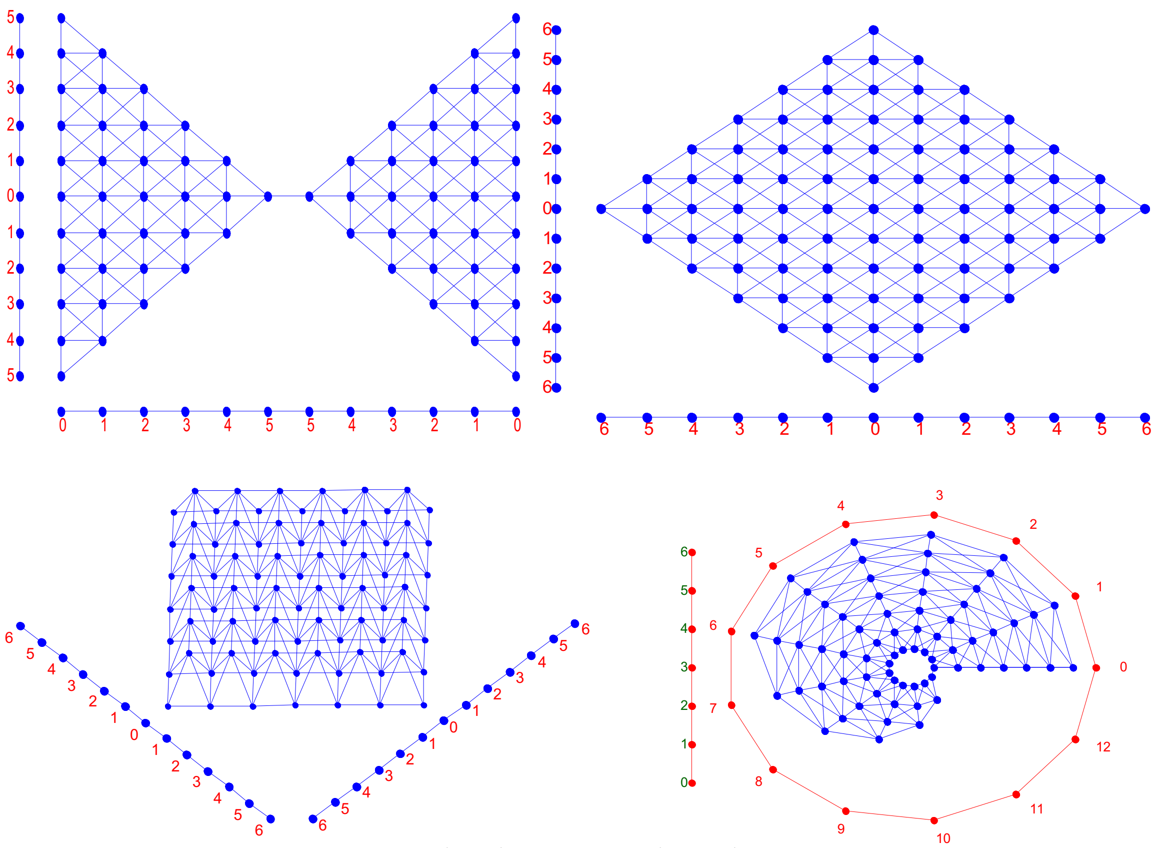

**Fig. 6** Triangular strong Cartesian products

### 4.4. Triangular Cartesian direct graph products of type I

Using this graph product, it is possible to connect one of the diagonal elements in the panels. Two nodes $(U_k, V_l)$ and $(U_i, V_j)$ in the product graph are connected, if one of the following conditions are satisfied:

$$\text{if } \{[[[U_i = U_k \text{ and } V_jV_l \in E(K)] \text{ or } [U_iU_k \in E(H) \text{ and } V_j = V_l]] \text{ and } [W_{i,j}, W_{k,l} = W_{max}]] \text{ or } [[U_iU_k \in E(K) \text{ and } V_jV_l \in E(H)] \text{ and } [W_{i,j}, W_{k,l} \leq W_{max} \text{ and } W_{i,j} = W_{k,l}]]\}$$

Four examples of grids generated by these relationships are illustrated in Fig. 7. For each configuration the corresponding weighted generators are also shown.

### 4.5. Triangular Cartesian direct graph products of type II

By using this graph product, it is possible to connect one of the diagonal elements in each panel. For two nodes $(U_k, V_l)$ and $(U_i, V_j)$ in the product graph, the connectedness is possible, if one of the following conditions are satisfied:

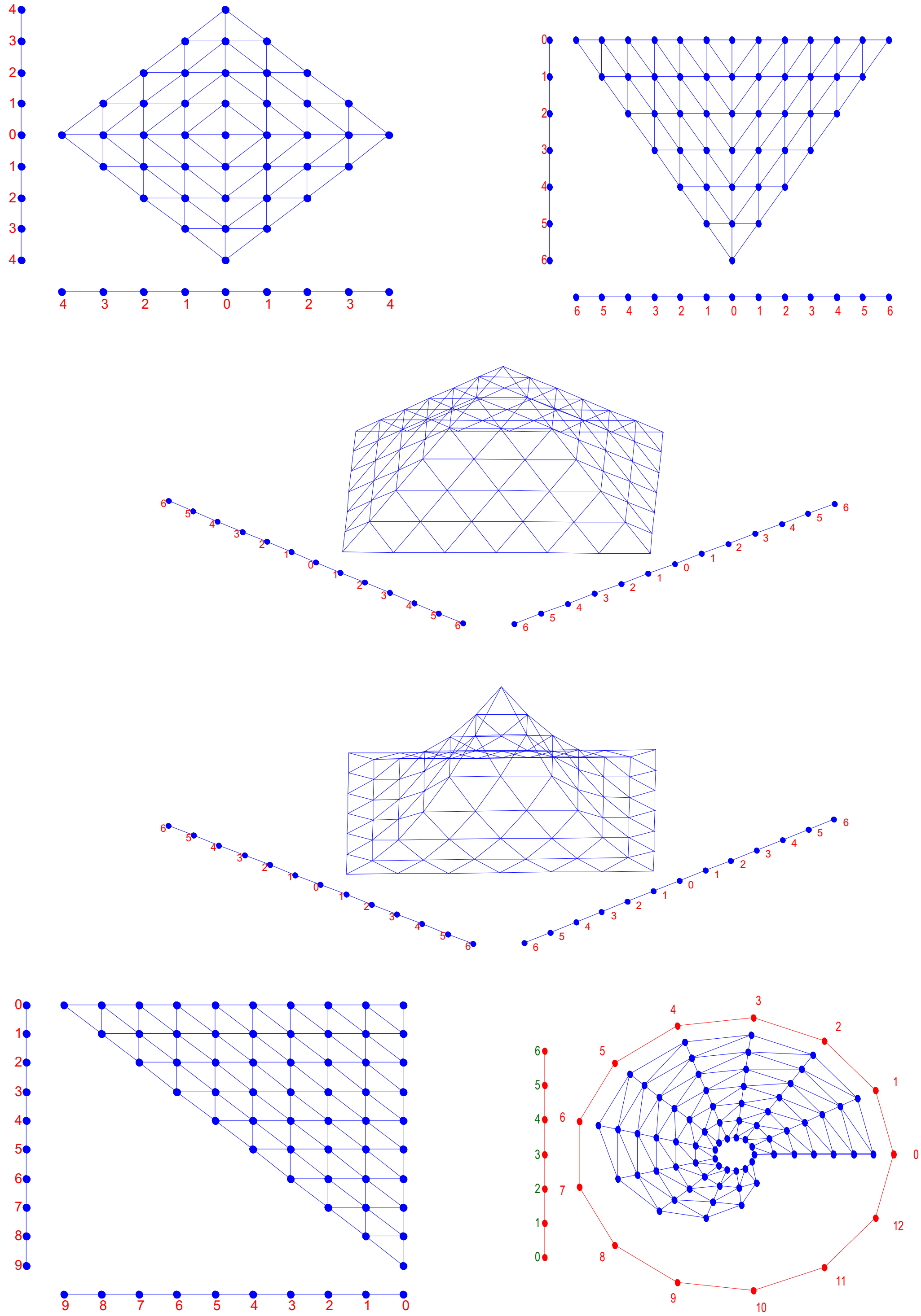

**Fig. 7** Triangular Cartesian direct products of type I

*If* $\{[[[U_i = U_k$ *and* $V_jV_l \in E(K)]$ *or* $[U_iU_k \in E(H)$ *and* $V_j = V_l]]$ *and* $[W_{i,j}, W_{k,l} = W_{max}]]$
*or* $[[U_iU_k \in E(K)$ *and* $V_jV_l \in E(H)]$ *and* $[W_{i,j}, W_{k,l} = W_{max}]]$
*or* $[[U_iU_k \in E(K)$ *and* $V_jV_l \in E(H)]$ *and* $[W_{i,j}, W_{k,l} \leq W_{max}$ and $W_{i,j} = W_{k,l} + 2]]\}$

Four examples of grids generated by these relationships are shown in Fig. 8.

*4.6. Triangular direct graph products*

This graph product is similar to the classic direct product with the difference that the connected nodes in the product graph will have the weight equal or smaller than $W_{max}$. Two nodes $(U_k, V_l)$ and $(U_i, V_j)$ in the product graph will be connected if one of the following conditions are satisfied:

*If* $\{[[[U_iU_k \in E(K)$ *and* $V_jV_l \in E(H)]$ *and* $[W_{i,j}, W_{k,l} \leq W_{max}$ *and* $W_{i,j} = W_{k,l} + 2$ *or* $W_{i,j} = W_{k,l}]]]$
*or* $[[U_i = U_k$ *and* $V_jV_l \in E(K)$ *or* $U_iU_k \in E(H)$ *and* $V_j = V_l]$ *and* $[W_{i,j}, W_{k,l} = W_{max}]]\}$

Four examples of grids generated by these relationships are shown in Fig. 9.

## 5. Definition of the circular graph products

In the field of space structures, domes are of special importance. From the structural point of view domes are always similar to solid circular shaped models. For configuring solid circles using graph products, one should use the transformation of product graph of two paths or the graph product of a path and a cycle should be employed. Another graph for covering the central part can be added. In this section, a new graph product is defined to configure solid circular shaped models.

*5.1. Circular Cartesian graph products*

In this product weights are assigned to the nodes of the generators, and the condiions and definitions of this product are based on these weights. In this product, the conditions are merely provided for the adjacency of the vertical and

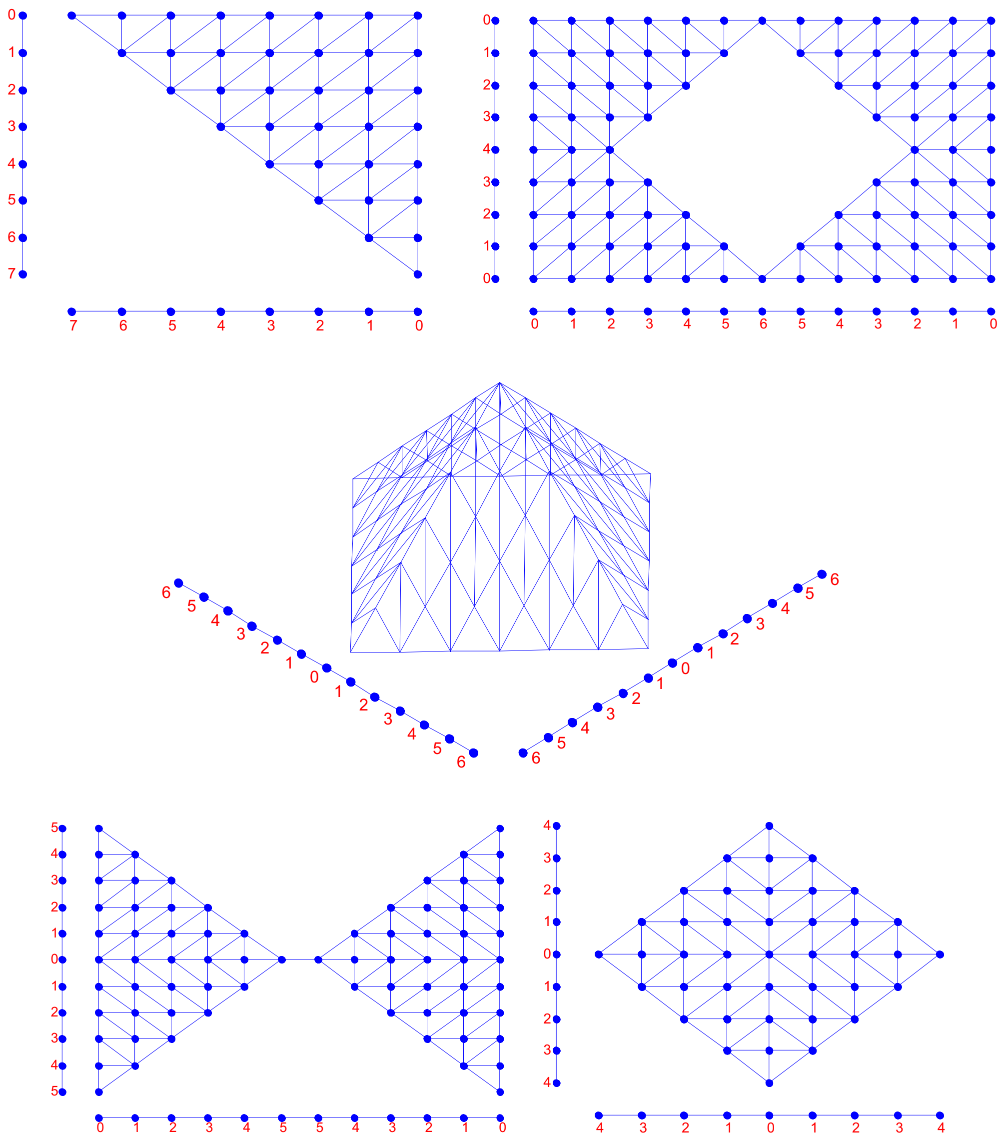


**Fig. 8** Triangular Cartesian direct products of type II

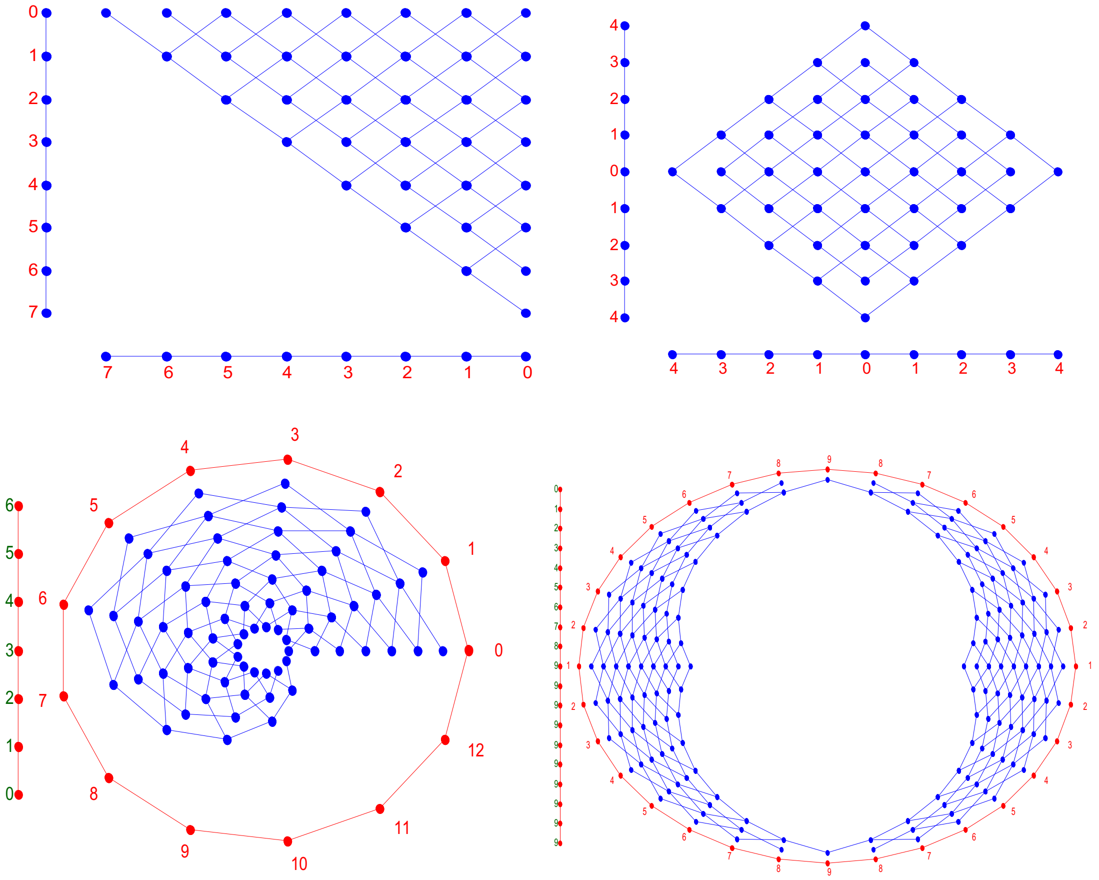

**Fig. 9** Triangular Cartesian direct product of type II

horizantel elements. Two nodes $(U_k, V_l)$ and $(U_i, V_j)$ in the product graph, with the weights $W_i$ , $W_j$ , $W_k$ and $W_l$ , are adjacent if the following conditions are fulfilled:

$If\ \{[[[U_iU_k\ and\ V_jV_l \in E(K)]\ or\ [U_iU_k \in E(H)\ and\ V_j = V_l]]\ and\ [W_{i,j}, W_{k,l} \leq 1]]$
$or\ [[V_jV_l \in E(K)]\ and\ [W_j > 1\ and\ i = 1]]\}$

Some examples of this product are demonstrated in Fig. 10.

*5.2. Circular strong Cartesian graph products*

In this product the nodes of the generators are weighted and conditions are defined based on these weights. Using this product two nodes $(U_k, V_l)$ and $(U_i, V_j)$ in the product graph having the weights $W_i$ , $W_j$ , $W_k$ and $W_l$ will be connected if the following conditions are fulfilled:

$If\ \{[[[U_iU_k\ and\ V_jV_l \in E(K)]\ or\ [U_iU_k \in E(H)\ and\ V_j = V_l]$
$or\ [U_iU_k \in E(K)\ and\ V_j = V_l \in E(H)]]\ and\ [W_j , W_l \leq 1]]$
$or\ [[V_jV_l \in E(K)]\ and\ [W_j > 1\ and\ i = 1]]\}$

Some examples of this product are illustrated in Fig. 11.

*5.3. Circular direct graph product models*

In this product only the diagonal elements will be connected. The nodes of the generators are weighted and conditions are defined based on the weights.

Using this product two nodes $U(i,j)$ and $V(k,l)$ in the product graph with weights $W_i$ , $W_j$ , $W_k$ and $W_l$ assigned to them, are connected if the following conditions are fulfilled:

$If\ \{[[U_iU_k \in E(K)\ and\ V_j = V_l \in E(H)]\ and\ [W_j , W_l \leq 1]]$
$or\ [[V_jV_l \in E(K)]\ and\ [W_j > 1\ and\ i = 1]]\}$

Some examples of this product are demonstrated in Fig. 12.

## 6. Cut out in graph products

Utilizing all the previous prodcuts, the product graphs have continuous and regular shapes and models and all of the nodes in the inner panels are connected. However, some models are in forms where there exist some inner panels as hollow and some nodes have no connections with the other nodes. On the other hand, the elements connected to some nodes are not present in the model and must be eliminated from the continuous model. In this section a new graph product for eliminating an optional node or nodes with the connected elements (stars) is defined. Using this product, all the nodes in the product graph which have equal and larger than an specfied weight will have no connections with the other nodes and elements.

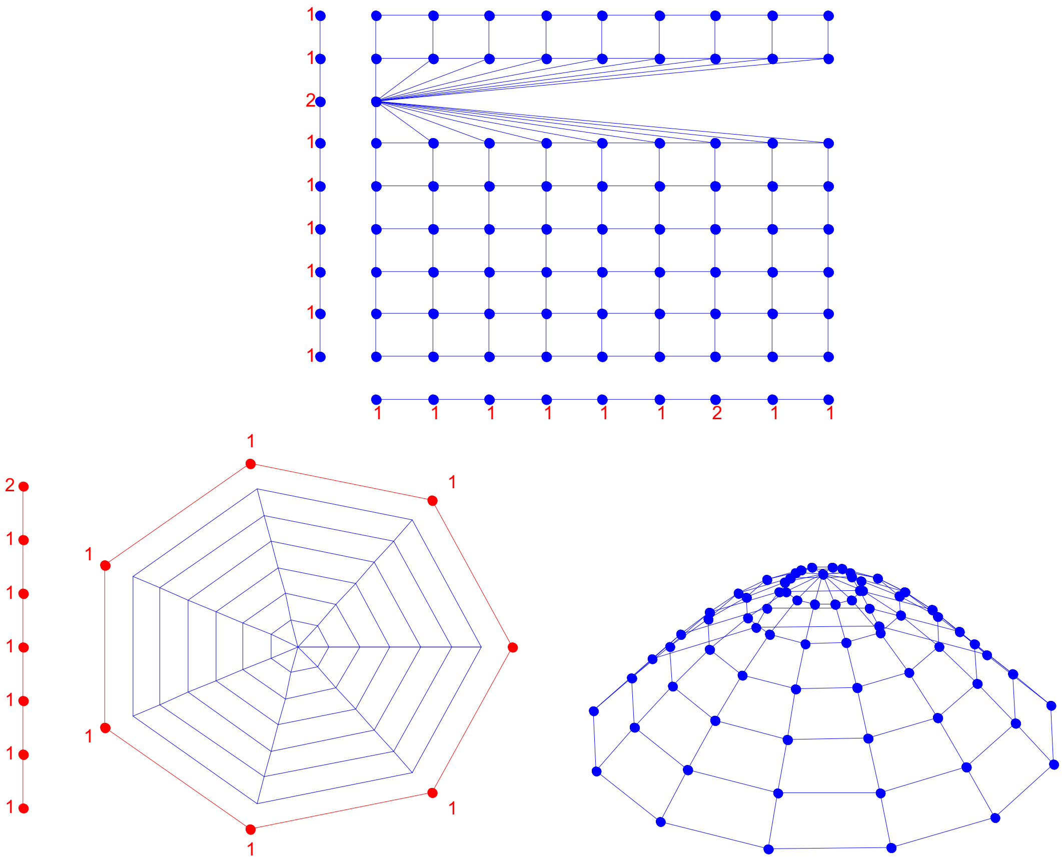


**Fig. 10** Some circular Cartesian products

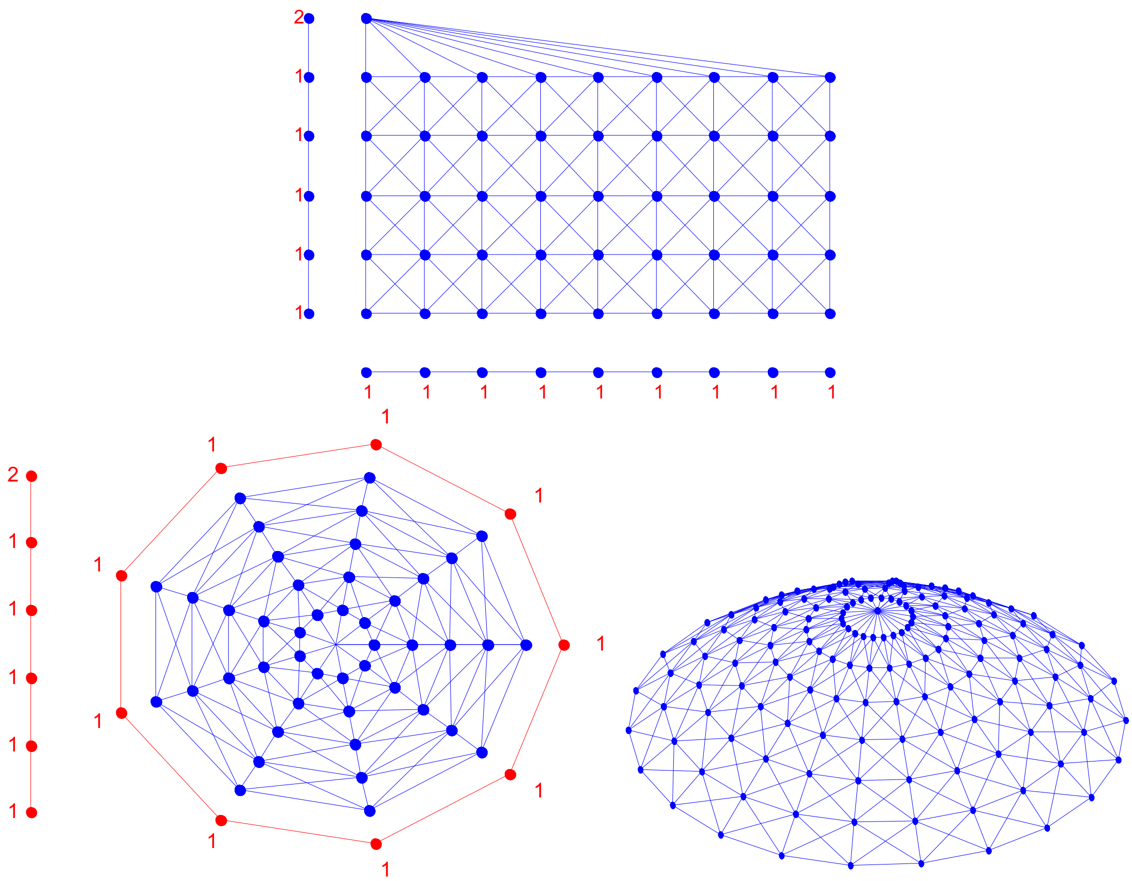


**Fig. 11** Circular strong Cartesian product models

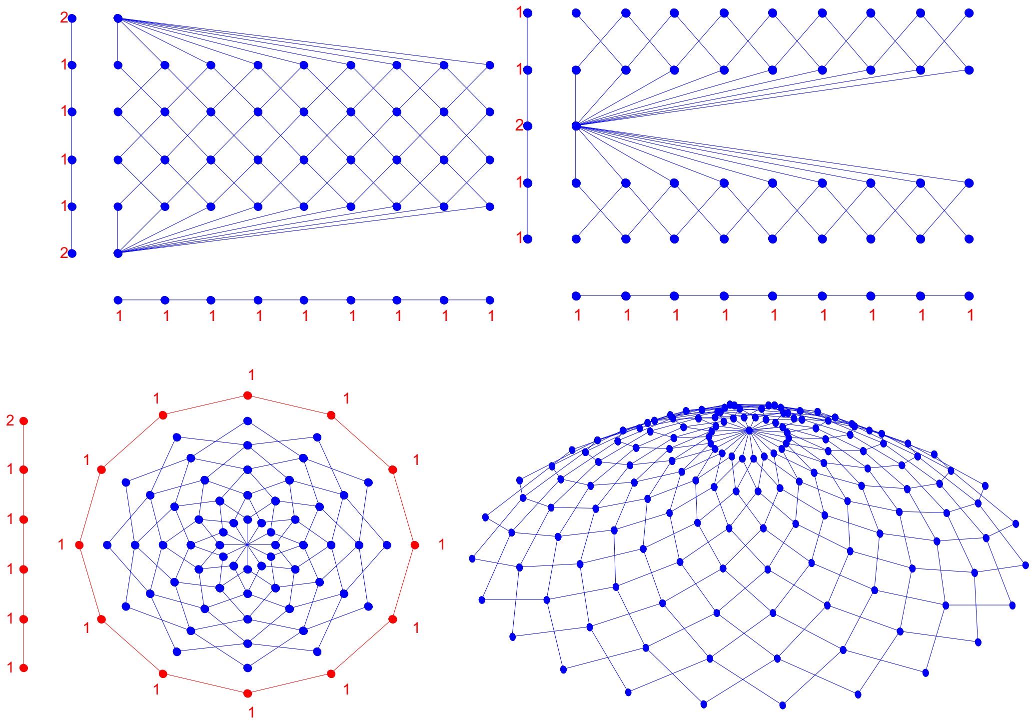

**Fig. 12** Circular direct product models

*6.1. Cut outs in Cartesian graph product models*

This graph product provides the conditions for connecting vertical and horizontal elements according to the weights of the generators. Using this product the nodes U(i,j) and V(k,l) in the product graph with weights Wi, Wj, Wk and Wl assigned to them, are connected if the following conditions are fulfilled:

*If* $\{[[U_iU_k \text{ and } V_jV_l \in E(K)] \text{ or } [U_iU_k \in E(H) \text{ and } V_j = V_l]]$ $\text{and } [W_i, W_j, W_k, W_l \leq 1] \text{ or } [W_i \neq W_j \text{ and } W_k \neq W_l]]\}$

Examples of this product are demonstrated in Fig. 13.

*6.2. Cut out in strong Cartesian graph products*

This product provides the conditions for connecting vertical, horizontal and diagonal elements according to the weights of the generators. Using this product the nodes $U(i,j)$ and $V(k,l)$ in the product graph with weights $W_i$, $W_j$, $W_k$ and $W_l$ assigned to them, will be connected if the following conditions are fulfilled:

*If* $\{[[U_i = U_k \text{ and } V_jV_l \in E(K)] \text{ or } [U_iU_k \in E(H) \text{ and } V_j = V_l]$ or $[U_iU_k \in E(K) \text{ and } V_jV_l \in E(H)]]$ $\text{and } [[W_i, W_j, W_k, W_l \leq 1] \text{ or } [W_i \neq W_j \text{ and } W_k \neq W_l]]\}$

Examples of this product are demonstrated in Fig. 14.

*6.3. Cut outs in direct graph products*

In this product, conditions consist of direct graph product conditions and a new additional condition for deleting some nodes with special weights. In this product only the diagonal elements will be connected. Using this product the nodes $U(i,j)$ and $V(k,l)$ in the product graph with the weights $W_i$, $W_j$, $W_k$ and $W_l$ assigned to them, will be connected if the following conditions are fulfilled:

*If* $\{[[U_iU_k \in E(K)] \text{ and } [V_jV_l \in E(H)]]$ $\text{and } [[W_i, W_j, W_k, W_l \leq 1] \text{ or } [W_i \neq W_j \text{ and } W_k \neq W_l]]\}$

An example of this product is demonstrated in Fig. 15.

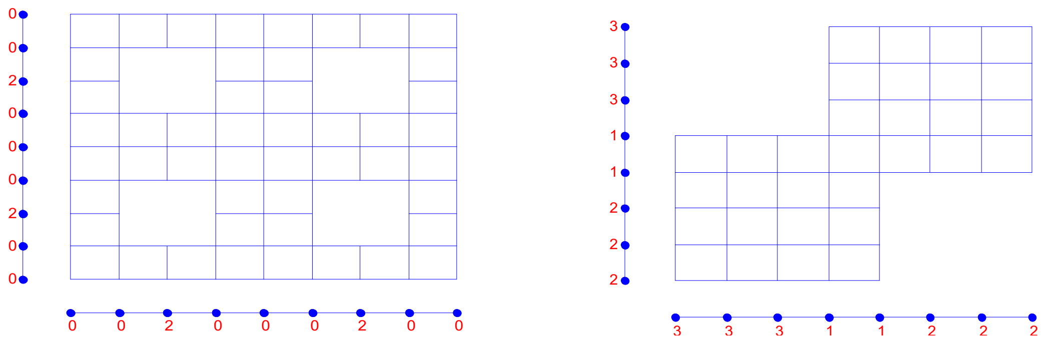

**Fig. 13** Cut out in Cartesian product models

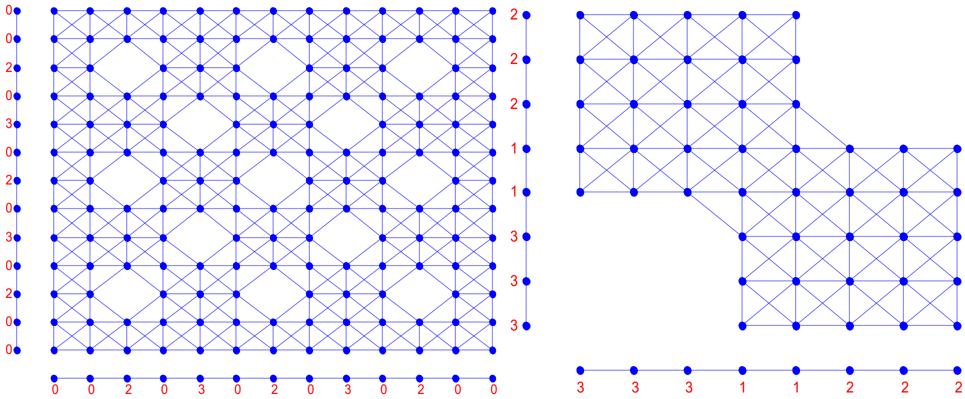


**Fig. 14** Cut outs in strong Cartesian product models

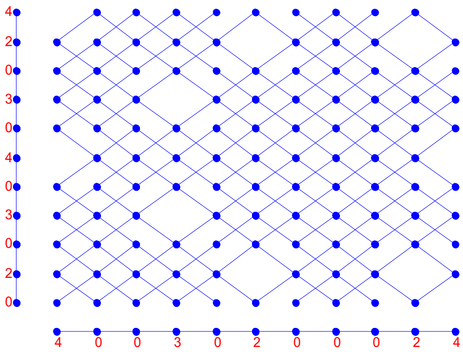


**Fig. 15** Cut outs in a direct product model

## 7. Concluding remarks

Using the new triangular graph product it becomes possible to generate triangular and a wide range of polygonal shaped configurations. It is also possible to configure triangular, trapezoid, parallelogram and many combinations of triangular and rectangular shaped models. Employing triangular graph products, the formation of differnt configurations based on a simple algebra and graph theory become feasible. Circular graph products make it possible to generate the circular shapes which are the models of some space structures like domes.

The use of graph products reduces the number of operations required for the formation of large-scale structural model and makes generation much easier and less costly in the computer. This is because in place of the data for the entire model only the information of two much smaller subgraphs (generators) is needed to be stored.

**Acknowledgement:** The first author is grateful to Iran National Science Foundation for the support.

## References

[1] Nooshin H. Algebraic representation and processing of structural configurations, Computers and Structures, 1975, Vol. 5, pp. 119–130.

[2] Nooshin H. Formex Configuration Processing in Structural Engineering, Elsevier Applied Science Publishers, London, 1984.

[3] Nooshin H, Disney P, Yamamoto C. Formian, U.K.: Multi-Science Publishers. J. Rebielak, 1995.

[4] Nooshin H, and Disney P. Formex configuration Processing I, II, III, International Journal of Space Structures, 2000 Vol. 15, pp. 1–52; 2001, Vol. 16, pp. 1–56; 2002, Vol. 17, pp. 1–50.

[5] Behravesh A, Kaveh A, Nani M, Sabet S. A set theoretical approach to configuration processing, Computers and Structures, 1988, Vol. 30, pp. 1293–1302.

[6] Kaveh A. A graph theoretical approach to configuration processing, Computers and Structures, 1993, Vol. 48, pp. 357–363.

[7] Kaveh A. Structural Mechanics: Graph and Matrix Methods, Research Studies Press, 3rd edition, Somerset, UK, 2004

[8] Kaveh A, Koohestani K. Graph products for configuration processing of space structures, Computers and Structures, 2008, Vol. 86, pp. 1219-1236.

[9] Kaveh A, Nouri M. Weighted graph products for configuration, Processing of Planar and Space Structures, International Journal of Space Structures, 2009, Vol. 24, pp. 13-26.

[10] Thacker WC. A brief review of techniques for generating irregular computational grids, International Journal for Numerical Methods in Engineering, 1980, Vol. 15, pp. 1335–1341.

[11] Thomson JF, Warsi ZUA, Hastin CW. Numerical Grid Generation, Foundations and Applications, Elsevier, Amsterdam, 1985.

[12] Vladimir D. Liseikin, Grid Generation Methods, Series: Scientific Computation, Springer Verlag, 1999.

[13] Kaveh A, Rahami H. Block diagonalization of adjacency and Laplacian matrices for graph products; Applications in structural mechanics, International Journal for Numerical Methods in Engineering, 2006, Vol. 68, pp. 33–63.

[14] Kaveh A, Rahami H. A unified method for eigen-decomposition of graph products, Communications in Numerical Methods in Engineering, 2005, Vol. 21, pp. 377–388.

[15] Imrich W, Klavzar S. Product Graphs; Structure and Recognition, John Wiley, NY, 2000.

[16] Kaveh A. Optimal Structural Analysis, John Wiley, 2nd edition, Somerset, UK, 2006.

[17] Kaveh A, Beheshti S. Weighted triangular and circular graph products for configuration processing, Periodica Polytechnica-Civil Engineering, 2012, Vol. 56, pp. 63-71.

[18] Kaveh A, Alinejad B. Graph products with specified domains for configuration processing and formation of the adjacency matrices, Engineering Computations, 2010, Vol. 27, pp. 205-224.

[19] Kaveh A. Optimal Analysis of Structures by Concepts of Symmetry and Regularity, Springer Verlag, GmbH, Wien-New York, http://www.springer.com/materials/mechanics/book/978-3-7091-1564-0, to appear in April 2013.